Elementary Topology

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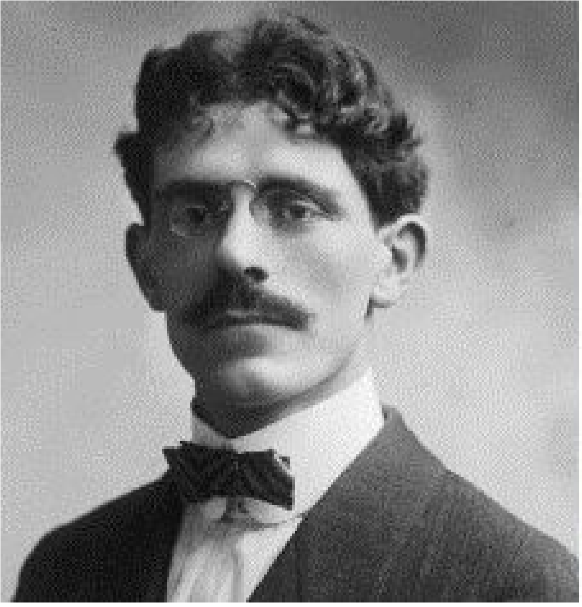


1. Metric Spaces and Point-Set Topology

Two fundamental and ubiquitous notions in Analysis are continuity and convergence. The word continuity comes from the latin *continere*, which can be translated as *hold together*. Intuitively, a function f from $X \subseteq \mathbb{R}$ to $\mathbb{R}$ is continuous if it does not produce any cut: when x varies slightly, also the image $f(x)$ varies slightly. The reader is certainly familiar with the following precise definition of continuity at a point x_0:

$$\text{For every } \varepsilon > 0, \text{ there exists } \delta > 0 \text{ such that } |f(x) - f(x_0)| < \varepsilon \text{ whenever } |x - x_0| < \delta.$$

Clearly, this definition relies on the fact that we can measure the "distance" between two points of the real line. A similar issue appears in the case of convergence. Intuitively, a sequence $\{x_n\}$ of real numbers converges to a limit x if the elements of the sequence become eventually as close to x as one wants. More precisely:

$$\text{For every } \varepsilon > 0, \text{ there exists } N \text{ such that } |x - x_n| < \varepsilon \text{ for each } n \geq N.$$

Once again, we see that the definition above requires some sort of "distance". It would certainly be interesting to generalize the notions of continuity and convergence to "spaces" different from $\mathbb{R}$. For example, what could it mean for a function $f : \mathbb{R}^n \to \mathbb{R}^m$ to be continuous? Can we speak about convergence of sequences of functions or of sequences of sequences?

Rather than defining continuity and convergence (and many other interesting concepts) for a particular "space", it is certainly desirable (and more efficient) to look for a sufficiently general notion of "space" in which we can properly define them. But in the first place, as already noticed, it seems necessary to have a meaningful notion of "distance". This will lead us to introduce a general object called *metric space*: we will see that most of what the reader is already familiar with can be easily translated into this new setting. On the other hand, it will immediately appear that continuity and convergence (and many other concepts we are interested in) are in fact independent of the "distance" and can be further generalized in the setting of a *topological space* which is somehow the natural ambient where continuous functions can be defined. This object will be introduced in Chapter 2 and studied throughout the manuscript.

We now introduce the notion of metric space, which is nothing but a set endowed with a "distance function".

Definition 1.1 — Metric space. A metric, or distance, on a set X is a map $d\colon X \times X \to \mathbb{R}$ satisfying the following properties:

M1. $d(x,y) \geq 0$ for all $x,y \in X$, with equality if and only if $x = y$;
M2. $d(x,y) = d(y,x)$ for all $x,y \in X$;
M3. $d(x,z) \leq d(x,y) + d(y,z)$ for all $x,y,z \in X$.

A metric space is a pair (X,d), where X is a set and d a metric on X.

The property M3 is called *triangle inequality*, since when considering the usual distance in the plane it says that the length of one side of a triangle is at most the sum of the lengths of the other two sides. This distance can be immediately generalized:

■ **Example 1.1 — Euclidean metric.** Given $\boldsymbol{x} = (x_1,\ldots,x_n)$ and $\boldsymbol{y} = (y_1,\ldots,y_n)$ in the Euclidean n-space $\mathbb{R}^n$, we define the *Euclidean distance* as

$$d(\boldsymbol{x},\boldsymbol{y}) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}.$$

Note that $d(\boldsymbol{x},\boldsymbol{y}) = \|\boldsymbol{x}-\boldsymbol{y}\|$, where $\|\cdot\|$ is the usual Euclidean norm associated with the inner product $(\cdot,\cdot)$ in $\mathbb{R}^n$, i.e. $(\boldsymbol{x},\boldsymbol{y}) = \sum x_i y_i$ and $\|\boldsymbol{x}\| = \sqrt{(\boldsymbol{x},\boldsymbol{x})}$. The Euclidean distance clearly satisfies M1 and M2. Let us now show it satisfies M3 as well and so, as the name suggests, it is indeed a distance. Given $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in \mathbb{R}^n$, we have to show that $\|\boldsymbol{x}-\boldsymbol{z}\| \leq \|\boldsymbol{x}-\boldsymbol{y}\| + \|\boldsymbol{y}-\boldsymbol{z}\|$. Therefore, letting $\boldsymbol{u} = \boldsymbol{x}-\boldsymbol{y}$ and $\boldsymbol{v} = \boldsymbol{y}-\boldsymbol{z}$, M3 is equivalent to $\|\boldsymbol{u}+\boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$ and further to

$$0 \leq (\|\boldsymbol{u}\| + \|\boldsymbol{v}\|)^2 - \|\boldsymbol{u}+\boldsymbol{v}\|^2 = 2(\|\boldsymbol{u}\|\|\boldsymbol{v}\| - (\boldsymbol{u},\boldsymbol{v})).$$

It is then enough to show the *Cauchy-Schwarz inequality*:

$$\|\boldsymbol{u}\|\|\boldsymbol{v}\| \geq |(\boldsymbol{u},\boldsymbol{v})|.$$

For $\lambda \in \mathbb{R}$, we have

$$0 \leq (\boldsymbol{u}+\lambda\boldsymbol{v},\boldsymbol{u}+\lambda\boldsymbol{v}) = (\boldsymbol{u},\boldsymbol{u}) + 2\lambda(\boldsymbol{u},\boldsymbol{v}) + \lambda^2(\boldsymbol{v},\boldsymbol{v}) = c + b\lambda + a\lambda^2 = f(\lambda).$$

Since the real-valued quadratic function f is non-negative, it must be $b^2 - 4ac \leq 0$, i.e. $4(\boldsymbol{u},\boldsymbol{v})^2 - 4(\boldsymbol{u},\boldsymbol{u})(\boldsymbol{v},\boldsymbol{v}) \leq 0$ from which the Cauchy-Schwarz inequality follows.

Therefore, $\mathbb{R}^n$ together with the Euclidean distance is a metric space. We will see later a generalization of this example. ■

■ **Example 1.2 — Discrete metric.** Let X be a set and for $x,y \in X$ define

$$d(x,y) = \begin{cases} 0 & \text{if } x = y; \\ 1 & \text{otherwise.} \end{cases} \tag{1.1}$$

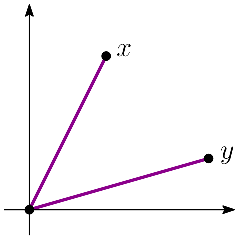
It is easy to see that (X,d) is a metric space, called *discrete*. ■

■ **Example 1.3 — SNCF metric.** Everyone who has ever tried to take a train from Nantes to Lyon has surely noticed that a stop in Paris is compulsory. This leads us to define a metric on $\mathbb{C}$, the *SNCF metric*[1], as follows:

$$d(x,y) = \begin{cases} |x-y| & \text{if } \operatorname{Arg}(x) = \operatorname{Arg}(y) \pmod{2\pi}; \\ |x| + |y| & \text{otherwise.} \end{cases} \tag{1.2}$$

Note that $|\cdot|$ is the usual absolute value of a complex number. In our interpretation, Paris is the origin of the complex plane.

[1] SNCF is the acronym for Société nationale des chemins de fer français.



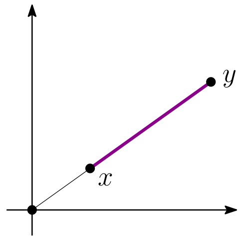


Figure 1.1: The SNCF metric or "all trains lead to Paris".

Let us check d is indeed a metric on $\mathbb{C}$. The only non-trivial verification is M3. Therefore, let $x,y,z \in \mathbb{C}$. Suppose first $\operatorname{Arg}(x) = \operatorname{Arg}(z) \pmod{2\pi}$. If in addition $\operatorname{Arg}(y) = \operatorname{Arg}(z) \pmod{2\pi}$, then

$$d(x,z) = |x-z| \leq |x-y| + |y-z| = d(x,y) + d(y,z).$$

Otherwise, we have

$$d(x,z) = |x-z| \leq |x| + |z| \leq (|x|+|y|) + (|y|+|z|) = d(x,y) + d(y,z).$$

Suppose now $\operatorname{Arg}(x) \neq \operatorname{Arg}(z) \pmod{2\pi}$. If in addition $\operatorname{Arg}(y) = \operatorname{Arg}(x) \pmod{2\pi}$, then

$$d(x,z) = |x| + |z| \leq |x-y+y| + |z| \leq |x-y| + |y| + |z| = d(x,y) + d(y,z).$$

Note that, by symmetry, the same holds if $\operatorname{Arg}(y) = \operatorname{Arg}(z) \pmod{2\pi}$. Therefore, assume $\operatorname{Arg}(y) \neq \operatorname{Arg}(x) \pmod{2\pi}$ and $\operatorname{Arg}(y) \neq \operatorname{Arg}(z) \pmod{2\pi}$, thus obtaining

$$d(x,z) = |x| + |z| \leq (|x|+|y|) + (|y|+|z|) = d(x,y) + d(y,z).$$

This shows that $\mathbb{C}$ together with the SNCF metric is a metric space. ■

Exercise 1.1 Let (X,d) be a metric space. Show that $d'(x,y) = \frac{d(x,y)}{1+d(x,y)}$ defines a metric on X.
Enter ■

Exercise 1.2 Let G be a connected graph. The distance $d(v_i,v_j)$ from the vertex v_i to the vertex v_j of G is the minimum length of path between v_i and v_j. Show that (G,d) is a metric space. ■

Exercise 1.3 For each integer $n \neq 0$ and prime number p, let $v_p(n) = \max\{r : p^r \mid n\}$. For each $0 \neq a/b \in \mathbb{Q}$, let $v_p(a/b) = v_p(a) - v_p(b)$. Show first that $v_p(r)$ is well-defined for each rational number $r \neq 0$. Finally, show that

$$d(x,y) = \begin{cases} p^{-v_p(x-y)} & \text{if } x \neq y; \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

defines a metric on $\mathbb{Q}$ such that $d(x,z) \leq \max\{d(x,y), d(y,z)\}$ for all $x,y,z \in \mathbb{Q}$. ■

It turns out that the metric defined in Example 1.1 comes from a very general construction related to the notion of norm:

Definition 1.2 — Normed space. Given a vector space E over the field $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, a norm on E is a map $\|\cdot\|\colon E \to \mathbb{R}$ satisfying the following properties:

N1. $\|x\| \geq 0$ for every $x \in E$, with equality if and only if $x = 0$;
N2. $\|\lambda x\| = |\lambda|\|x\|$ for every $\lambda \in \mathbb{F}$ and $x \in E$;
N3. $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in E$.

A normed space is a pair $(E, \|\cdot\|)$, where E is a vector space and $\|\cdot\|$ a norm on E.

An easy but important observation is that a normed space $(E, \|\cdot\|)$ is in fact a metric space: just set $d(x,y) = \|x-y\|$, for all $x, y \in E$. The vector space $\mathbb{R}^n$ with the Euclidean metric is an example of this kind.

Note that a metric on a normed space defined as above is *homogeneous*, i.e. $d(ax, ay) = |a|d(x,y)$ for every $a \in \mathbb{F}$ and $x, y \in E$, and *translation invariant*, i.e. $d(x+z, y+z) = d(x,y)$ for every $x, y, z \in E$. This shows that a metric on a vector space does not necessarily come from a norm as described above. For example, consider the discrete metric or the SNCF metric.

Exercise 1.4 Let E be a vector space over the field $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Show that if (E,d) is a metric space such that d is homogeneous and translation invariant, then $\|x\| = d(x,0)$ defines a norm on E. Enter ■

Let us see some examples of normed spaces. We begin with two norms defined on the finite-dimensional vector space $\mathbb{R}^n$.

■ **Example 1.4 — Max-norm.** Define the function $\|\cdot\|_\infty\colon \mathbb{R}^n \to \mathbb{R}$ by

$$\|\boldsymbol{x}\|_\infty = \max\{|x_1|, \ldots, |x_n|\}.$$

It clearly defines a norm on $\mathbb{R}^n$. ■

■ **Example 1.5 — p-norm.** Given a real number $p \geq 1$, we define the function $\|\cdot\|_p\colon \mathbb{R}^n \to \mathbb{R}$ by

$$\|\boldsymbol{x}\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}.$$

The case $p = 1$ gives rise to the so-called *Manhattan norm* and verifying N1 to N3 is immediate.

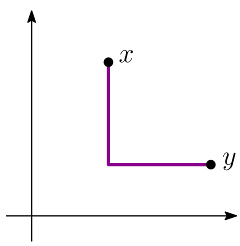


Figure 1.2: The Manhattan metric in $\mathbb{R}^2$.

For $p = 2$, we obtain the usual Euclidean norm (see Example 1.1) and we have already seen that M3 follows by the Cauchy-Schwarz inequality.

Let us now verify the norm properties in general. In fact, it is easy to check that the p-norm satisfies N1 and N2. In order to show N3, observe first that the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(t) = |t|^p$ is convex (see Exercise 1.5), i.e. for each $x, y \in \mathbb{R}$ and $\lambda \in [0,1]$, we have

$f(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y)$. Therefore, for every $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and $\lambda \in [0,1]$, we have

$$\|\lambda \boldsymbol{x}+(1-\lambda)\boldsymbol{y}\|_p^p = \sum_{i=1}^n |\lambda x_i+(1-\lambda)y_i|^p \leq \sum_{i=1}^n \lambda|x_i|^p+(1-\lambda)|y_i|^p = \lambda\|\boldsymbol{x}\|_p^p+(1-\lambda)\|\boldsymbol{y}\|_p^p.$$

This implies that if $\|\boldsymbol{x}\|_p = \|\boldsymbol{y}\|_p = 1$, then $\|\lambda\boldsymbol{x}+(1-\lambda)\boldsymbol{y}\|_p \leq 1$. Since N3 is trivial if one of $\boldsymbol{x}$ and $\boldsymbol{y}$ is $\mathbf{0}$, we may assume this is not the case. Therefore, $\boldsymbol{x}/\|\boldsymbol{x}\|_p$ and $\boldsymbol{y}/\|\boldsymbol{y}\|_p$ have both p-norm 1 and, by the previous inequality, we have

$$\frac{\|\boldsymbol{x}+\boldsymbol{y}\|_p}{\|\boldsymbol{x}\|_p+\|\boldsymbol{y}\|_p} = \left\| \frac{\|\boldsymbol{x}\|_p}{\|\boldsymbol{x}\|_p+\|\boldsymbol{y}\|_p}\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_p} + \frac{\|\boldsymbol{y}\|_p}{\|\boldsymbol{x}\|_p+\|\boldsymbol{y}\|_p}\frac{\boldsymbol{y}}{\|\boldsymbol{y}\|_p}\right\|_p \leq 1.$$

This is usually called the *Minkowski's inequality*. ■

Exercise 1.5 Show that the function $f\colon \mathbb{R}\to\mathbb{R}$ given by $f(t) = |t|^p$ is convex as follows:

- Show that $f\colon \mathbb{R}\to\mathbb{R}$ such that $f(t) = |t|$ is convex.
- Show that if $f\colon \mathbb{R}\to\mathbb{R}$ is convex and $g\colon \mathbb{R}\to\mathbb{R}$ is convex and non-decreasing, then $g\circ f$ is convex.

Enter ■

We now concentrate on an infinite-dimensional vector space over $\mathbb{R}$: the vector space of all continuous functions $f\colon [0,1]\to\mathbb{R}$.

■ **Example 1.6 — Sup-norm.** Let $C^0([0,1],\mathbb{R})$ be the vector space of all continuous functions $f\colon [0,1]\to\mathbb{R}$. We define the sup-norm $\|\cdot\|_\infty$ on $C^0([0,1],\mathbb{R})$ by

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|.$$

Weierstrass' Theorem implies $\|\cdot\|_\infty$ is well-defined. N1 and N2 are an easy check. Moreover, for each $x\in[0,1]$, we have $|f(x)+g(x)| \leq |f(x)|+|g(x)| \leq \|f\|_\infty+\|g\|_\infty$ and so N3 holds as well. We refer to the corresponding metric as *uniform metric* and the reason will appear in Example 1.12. ■

■ **Example 1.7 — L^1-norm.** Consider again the vector space $C^0([0,1],\mathbb{R})$ of all continuous functions $f\colon [0,1]\to\mathbb{R}$. We define the L^1-norm $\|\cdot\|_{L^1([0,1])}$ on $C^0([0,1],\mathbb{R})$ by

$$\|f\|_{L^1([0,1])} = \int_0^1 |f(t)|\,\mathrm{d}t.$$

It is easy to check that N2 and N3 hold. Consider now N1. Suppose that $\|f\|_{L^1([0,1])} = 0$, i.e. $\int_0^1 |f(t)|\,\mathrm{d}t = 0$, but $|f|$ is not identically 0. This means that $|f(t_0)| > 0$, for some $t_0\in[0,1]$, and the continuity of $|f|$ implies there exists r such that $|f(t)| \geq \frac{|f(t_0)|}{2}$ for each $t\in[t_0-r, t_0+r]$. Therefore,

$$0 = \int_0^1 |f(t)|\,\mathrm{d}t \geq \int_{t_0-r}^{t_0+r} |f(t)|\,\mathrm{d}t \geq \int_{t_0-r}^{t_0+r} \frac{|f(t_0)|}{2}\,\mathrm{d}t = r|f(t_0)| > 0,$$

a contradiction. We refer to the corresponding metric as *integral metric*. ■

The reader is certainly familiar with the notion of open interval in $\mathbb{R}$. Having a metric at disposal, it can be easily generalized as follows:

Definition 1.3 — Open ball. Let (X,d) be a metric space, $x_0\in X$ and $r>0$. The open ball centered at x_0 with radius r is the set $B_r(x_0) = \{x\in X : d(x,x_0) < r\}$.

Similarly, it is natural to define an open set as a set such that, for each of its points, there exists an open ball containing the point and contained in the set:

Definition 1.4 — Open set. Let (X,d) be a metric space. A set $U \subset X$ is open if, for each $x \in U$, there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U$.

Note that, in the definition of open set, it is not the distance that plays a fundamental role, but rather the notion of "closeness" induced by the distance itself. This will become more evident by comparing Figure 1.3, Lemma 1.5 and Example 1.15.

Lemma 1.1 Every open ball in a metric space is open.

Proof. Let (X,d) be a metric space and consider the open ball centered at x_0 with radius r. For $x \in B_r(x_0)$, let $\varepsilon = r - d(x,x_0) > 0$. For each $y \in B_\varepsilon(x)$, we have

$$d(y,x_0) \leq d(y,x) + d(x,x_0) < \varepsilon + d(x,x_0) = r.$$

Therefore, $B_\varepsilon(x) \subset B_r(x_0)$, as claimed. ■

▪ **Example 1.8** Consider $\mathbb{R}$ with the Euclidean metric. The open balls correspond to the open bounded intervals (a,b). The intervals $(-\infty,b)$ and $(a,+\infty)$ are open as well as the complements of finite sets. Examples of non-open sets are $[a,b)$, $(a,b]$, $[a,b]$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}\setminus\mathbb{Q}$. ▪

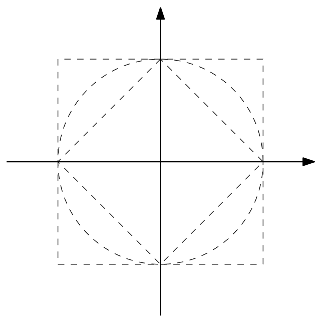


Figure 1.3: The open balls in $\mathbb{R}^2$ centered at $\mathbf{0}$ with radius 1 for the Manhattan metric, the Euclidean metric and the metric coming from the max-norm.

The following is an extremely important property of the collection of open sets:

Proposition 1.1 Let (X,d) be a metric space and $\mathscr{F}$ be the family of open sets of (X,d). The following hold:

(i). $\varnothing$ and X are in $\mathscr{F}$;
(ii). The union of the elements of any subfamily of $\mathscr{F}$ is in $\mathscr{F}$;
(iii). The intersection of the elements of any finite subfamily of $\mathscr{F}$ is in $\mathscr{F}$.

Proof. (i) is obvious.

(ii) Let $\mathscr{F}' \subset \mathscr{F}$ be any subfamily of open sets in (X,d) and let $x \in \bigcup\{F : F \in \mathscr{F}'\}$. Clearly, $x \in F_0$ for some $F_0 \in \mathscr{F}'$ and since F_0 is open, there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset F_0 \subset \bigcup\{F : F \in \mathscr{F}'\}$. This means that $\bigcup\{F : F \in \mathscr{F}'\}$ is open.

(iii) It is enough to show that for any two open sets F_1 and F_2, the intersection $F_1 \cap F_2$ is open. Therefore, let $x \in F_1 \cap F_2$. Since F_1 and F_2 are both open, there exist $\varepsilon_1, \varepsilon_2 > 0$ such that $B_{\varepsilon_1}(x) \subset F_1$ and $B_{\varepsilon_2}(x) \subset F_2$. Taking $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, we have $B_\varepsilon(x) \subset F_1 \cap F_2$. ■

R The intersection of the elements of any subfamily of open sets is not necessarily open. Indeed, consider $\mathbb{R}$ with the Euclidean metric and let I_n be the open interval $(-\frac{1}{n}, \frac{1}{n})$. We have $\bigcap I_n = \{0\}$.

■ **Example 1.9** All subsets of a discrete metric space (X,d) are open. Indeed, for each $S \subset X$, we have $S = \bigcup_{x \in S}\{x\} = \bigcup_{x \in S} B_1(x)$. ■

Lemma 1.2 Every open set in a metric space (X,d) is a union of open balls.

Proof. Let U be an open set. For each $x \in U$, there exists ε_x such that $B_{\varepsilon_x}(x) \subset U$. Therefore, $\bigcup_{x \in U} B_{\varepsilon_x}(x) \subset U$. The other inclusion is obvious. ■

Many observations can be concisely expressed in terms of neighbourhoods:

Definition 1.5 — Neighbourhood. Let (X,d) be a metric space and let $x \in X$. A subset N of X is a neighbourhood of x if there exists an open subset U of X such that $x \in U \subset N$.

Proposition 1.2 Let (X,d) be a metric space and let $x \in X$. The following hold:

(i). A subset N of X is a neighbourhood of x if and only if there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset N$;

(ii). If N_1 and N_2 are neighbourhoods of x, then $N_1 \cap N_2$ is a neighbourhood of x as well;

(iii). A subset U of X is open if and only if U is a neighbourhood of each of its points.

Proof. (i) If there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset N$, then N is a neighbourhood of x by Lemma 1.1. Conversely, if N is a neighbourhood of x, there exists an open subset U of X such that $x \in U \subset N$. Since U is open, there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U \subset N$.

(ii) is obvious.

(iii) If U is open, then it is clearly a neighbourhood of each of its points. Conversely, let U be a subset of X which is a neighbourhood of each of its points. This means that, for each $y \in U$, there exists an open subset U_y of X such that $y \in U_y \subset U$. Since $U = \bigcup_{y \in U} U_y$, Proposition 1.1 implies that U is open. ■

Exercise 1.6 Show that every open set in $\mathbb{R}$ with the Euclidean metric is an at most countable union of disjoint open intervals. Enter ■

We now introduce another family of distinguished sets in a metric space:

Definition 1.6 — Closed set. Let (X,d) be a metric space. A set $U \subset X$ is closed if $X \setminus U$ is open.

Exercise 1.7 Let (X,d) be a metric space, $x_0 \in X$ and $r > 0$. The *closed ball* centered at x_0 with radius r is the set $\{x \in X : d(x,x_0) \leq r\}$. Show that it is closed. Enter ■

In analogy to Proposition 1.1, we have the following:

Proposition 1.3 Let (X,d) be a metric space and $\mathscr{F}$ be the family of closed sets of (X,d). The following hold:

(i). $\varnothing$ and X are in $\mathscr{F}$;

(ii). The intersection of the elements of any subfamily of $\mathscr{F}$ is in $\mathscr{F}$;

(iii). The union of the elements of any finite subfamily of $\mathscr{F}$ is in $\mathscr{F}$.

Proof. Simple application of De Morgan's laws. ■

Proposition 1.3 shows that the the notions of being open and being closed are not negations of each other: the empty set and the whole space are always both open and closed. It could also be that every set of a metric space is both open and closed (see Example 1.9). Moreover, in some cases, a set could be neither open nor closed: just consider $\mathbb{R}$ with the Euclidean metric and the set $[0,1)$ or the set of rational numbers $\mathbb{Q}$.

Having extended the notion of distance from $\mathbb{R}$ to a general metric space, we can finally do the same for convergence. Let us first recall the general notion of sequence.

Definition 1.7 — Sequence, subsequence. Let S be any set. A map from $\mathbb{N}$ to S is a sequence in S. Instead of $x\colon \mathbb{N} \to S$, we write $\{x_n\}$.

A sequence $\{y_k\}$ is a subsequence of $\{x_k\}$ if there exist $n_1 < n_2 < \cdots$ in $\mathbb{N}$ such that $y_k = x_{n_k}$ for each $k \in \mathbb{N}$.

Definition 1.8 — Convergence. Let (X,d) be a metric space. A sequence $\{x_n\}$ in X converges to $x \in X$ if, for each $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $d(x_n,x) < \varepsilon$ for every $n \geq n_\varepsilon$. The point x is the limit of $\{x_n\}$ and we write $x = \lim_{n\to\infty} x_n$ or $x_n \to x$.

In other words, $\{x_n\}$ converges to x if and only if, for each neighbourhood N of x, there exists $n_N \in \mathbb{N}$ such that $x_n \in N$ for every $n \geq n_N$. Indeed, suppose $\{x_n\}$ converges to x and let N be a neighbourhood of x. There exists $\varepsilon_N > 0$ such that $B_{\varepsilon_N}(x) \subset N$. On the other hand, by convergence, there exists $n_{\varepsilon_N} \in \mathbb{N}$ such that $x_n \in B_{\varepsilon_N}(x) \subset N$ for every $n \geq n_{\varepsilon_N}$. Conversely, suppose that for each neighbourhood N of x, there exists $n_N \in \mathbb{N}$ such that $x_n \in N$ for every $n \geq n_N$. Since every open ball centered at x_0 is a neighbourhood of x_0, the convergence immediately follows.

Therefore, convergence can be expressed in terms of open sets (or, more precisely, of neighbourhoods).

Another equivalent formulation is the following: $\{x_n\}$ converges to x if and only if $\{d(x_n,x)\}$ converges to 0 (in $\mathbb{R}$ with the Euclidean metric).

Lemma 1.3 — Uniqueness of limits. Let (X,d) be a metric space, $\{x_n\}$ a sequence in X and $x,x' \in X$ such that $\{x_n\}$ converges to both x and x'. We have $x = x'$.

Proof. Suppose that $x \neq x'$. This implies that $\varepsilon = d(x,x')/2 > 0$. By the definition of convergence there exist $n_1 \in \mathbb{N}$ such that $d(x_n,x) < \varepsilon$ for every $n \geq n_1$ and $n_2 \in \mathbb{N}$ such that $d(x_n,x') < \varepsilon$ for every $n \geq n_2$. Setting $n = \max\{n_1,n_2\}$, we have

$$d(x,x') \leq d(x,x_n) + d(x_n,x') < \varepsilon + \varepsilon = d(x,x'),$$

a contradiction. ■

■ **Example 1.10** Let (X,d) be a discrete metric space and let $\{x_n\}$ be a sequence in X convergent to $x \in X$. By definition, there exists $n_1 \in \mathbb{N}$ such that $d(x_n,x) < 1$ for every $n \geq n_1$ and so $x_n = x$ for every $n \geq n_1$. In other words, every convergent sequence in a discrete metric space is eventually constant. ■

The following example shows that, unsurprisingly, convergence may behave differently when considering different metrics.

■ **Example 1.11** Let us study the convergence of the sequence $\{e^{i/n}\}$ in $\mathbb{C}$ with the Euclidean metric and with the SNCF metric. We claim that the sequence converges to 1 in the former case, while it does not converge in the latter.

In the Euclidean metric we have

$$d(e^{i/n},1)=|e^{i/n}-1|=\left|\cos\frac{1}{n}-1+i\sin\frac{1}{n}\right|=\sqrt{\left(\cos\frac{1}{n}-1\right)^2+\sin^2\frac{1}{n}}\to 0.$$

Suppose now $\{e^{i/n}\}$ converges to some $z\in\mathbb{C}$ in the SNCF metric. If $z=0$, then $d(e^{i/n},0)=|e^{i/n}|=1$, a contradiction. Therefore, it must be $z\neq 0$. Clearly, $\operatorname{Arg}(e^{i/n})=\frac{1}{n}$ and so, except for at most one value of n, we have $d(e^{i/n},z)=|e^{i/n}|+|z|=1+|z|$, a contradiction again. ■

We can now characterize closed sets in terms of convergent sequences. This should explain the terminology adopted.

Lemma 1.4 Let (X,d) be a metric space. A set $U\subset X$ is closed if and only if for every convergent sequence $\{x_n\}$ with elements in U, its limit $x=\lim_{n\to\infty}x_n$ belongs to U.

Proof. Let $U\subset X$ be a closed set and suppose first $\{x_n\}$ is a convergent sequence with $x_n\in U$, for each n, but $x=\lim_{n\to\infty}x_n\notin U$. Since $N=X\setminus U$ is open, N is a neighbourhood of x. On the other hand, the convergence implies there exists $n_N\in\mathbb{N}$ such that $x_n\in N$ for every $n\geq n_N$, a contradiction.

Let now $U\subset X$ be a set such that for every convergent sequence $\{x_n\}$ with elements in U, its limit x belongs to U. Suppose that $X\setminus U$ is not open. This implies there exists $x\in X\setminus U$ such that $B_{1/n}(x)\cap U\neq\varnothing$, for every $n\in\mathbb{N}$. Therefore, we can build a sequence $\{x_n\}$ with $x_n\in B_{1/n}(x)\cap U$, for every n. Clearly, $x_n\to x$ and so, by assumption, $x\in U$, a contradiction. ■

Corollary 1.1 The following are equivalent for any two metric spaces (X,d) and (X,d'):

(i). A set U is open in (X,d) if and only if it is open in (X,d').
(ii). A sequence $\{x_n\}$ converges to x in (X,d) if and only if it converges to x in (X,d').

Proof. (i) $\Rightarrow$ (ii) By assumption, N is a neighbourhood in (X,d) if and only if it is a neighbourhood in (X,d') and the conclusion immediately follows.

(ii) $\Rightarrow$ (i) It is enough to show that a set V is closed in (X,d) if and only if it is closed in (X,d'). By Lemma 1.4, V is closed in (X,d) if and only if it contains all the limits of convergent sequences in (X,d) with elements in V. By assumption, these are exactly the limits of convergent sequences in (X,d') with elements in V. The conclusion follows. ■

■ **Example 1.12** We now provide a “natural” way to view the usual notion of uniform convergence of a sequence of continuous real-valued functions on the interval $[0,1]$: it is nothing but convergence in the metric space $C^0([0,1],\mathbb{R})$ equipped with the uniform metric (see Example 1.6). In other words, a sequence $\{f_n\}$ in $C^0([0,1],\mathbb{R})$ with the uniform metric converges to f (in $C^0([0,1],\mathbb{R})$) if and only if it converges uniformly to f on $[0,1]$.

Indeed, suppose first $\|f_n-f\|_\infty\to 0$. This means that, for every $\varepsilon>0$, there exists $n_\varepsilon\in\mathbb{N}$ such that $\sup_{t\in[0,1]}|f_n(t)-f(t)|=\|f_n-f\|_\infty<\varepsilon$ for each $n\geq n_\varepsilon$. In other words, for every $\varepsilon>0$, there exists $n_\varepsilon\in\mathbb{N}$ such that $|f_n(t)-f(t)|$ for each $n\geq n_\varepsilon$ and $t\in[0,1]$, i.e. $\{f_n\}$ converges uniformly to f on $[0,1]$.

Conversely, suppose $\{f_n\}$ converges uniformly to f on $[0,1]$ and let $\varepsilon>0$. Uniform convergence implies there exists $n_\varepsilon\in\mathbb{N}$ such that, for each $n\geq n_\varepsilon$ and $t\in[0,1]$, we have $|f_n(t)-f(t)|<\varepsilon/2$.

Therefore, for each $n \geq n_\varepsilon$, we have

$$\|f_n - f\|_\infty = \sup_{t\in[0,1]} |f_n(t) - f(t)| \leq \frac{\varepsilon}{2} < \varepsilon.$$

Recalling the well-known fact that a uniformly convergent sequence $\{f_n\}$ of continuous functions $f_n \colon [0,1] \to \mathbb{R}$ converges to a continuous function, we have that $f \in C^0([0,1],\mathbb{R})$. ■

Exercise 1.8 Show that the set $\{f \in C^0([0,1],\mathbb{R}) : f(x) = 0 \text{ for every } x \in [0,1]\}$ is a closed subset of $C^0([0,1],\mathbb{R})$ with the uniform metric. Enter ■

We have seen examples of sets equipped with different metrics. It is therefore natural to compare them:

Definition 1.9 — Equivalence of metrics. Two metrics d and d' on X are equivalent if there exist $c_1, c_2 > 0$ such that

$$c_1 d(x,y) \leq d'(x,y) \leq c_2 d(x,y), \quad \text{for every } x,y \in X.$$

R Note that being equivalent as defined in Definition 1.9 is indeed an equivalence relation.

In the following, we will see that in a metric space (and later in a topological space) many important properties like continuity, compactness and so on, can in fact be expressed in terms of open sets. In the context of metric spaces, Lemma 1.5 tells us that if any such property holds for (X,d), then it also holds for all the metrics on X equivalent to d.

Lemma 1.5 Let d and d' be two equivalent metrics on X. A set U is open in (X,d) if and only if it is open in (X,d').

Proof. Denote by $B_r(x_0)$ and $B'_r(x_0)$ the open balls in (X,d) and (X,d'), respectively. By assumption, there exist $c_1, c_2 > 0$ such that $c_1 d(x,y) \leq d'(x,y) \leq c_2 d(x,y)$, for every $x,y \in X$. This implies that, for each $x \in X$ and $\varepsilon > 0$, we have

$$B_{\varepsilon/c_1}(x) \supset B'_\varepsilon(x) \supset B_{\varepsilon/c_2}(x).$$

Therefore, if U is open in (X,d') then, for every $x \in U$, there exists $\varepsilon > 0$ such that $B_{\varepsilon/c_2}(x) \subset B'_\varepsilon(x) \subset U$ and so U is open in (X,d). Similarly, if U is open in (X,d) then, for every $x \in U$, there exists $\varepsilon > 0$ such that $B'_{c_1\varepsilon}(x) \subset B_\varepsilon(x) \subset U$ and so U is open in (X,d'). ■

■ **Example 1.13** The Euclidean metric and the discrete metric on $\mathbb{R}$ are not equivalent. This follows easily by Lemma 1.5, Example 1.9 and the fact that not every set in $\mathbb{R}$ with the Euclidean metric is open (just consider $[a,b)$ or $\mathbb{Q}$).

We can also argue as follows. The sequence $\{1/n\}$ converges in $\mathbb{R}$ with the Euclidean metric but it is not eventually constant and so it cannot converge in $\mathbb{R}$ with the discrete metric. Therefore, by Corollary 1.1 and Lemma 1.5, these two metrics are not equivalent. ■

■ **Example 1.14** The Euclidean metric and the SNCF metric on $\mathbb{C}$ are not equivalent. This follows easily by Example 1.11, the equivalence in Corollary 1.1 and Lemma 1.5. ■

■ **Example 1.15** The metrics coming from the p-norm and the max-norm on $\mathbb{R}^n$ are equivalent (see Figure 1.3). Indeed, it is easy to see that, for every $\boldsymbol{x} \in \mathbb{R}^n$ and distinct real numbers $p,q \geq 1$,

$$n^{-1/q}\|\boldsymbol{x}\|_q \leq \|\boldsymbol{x}\|_\infty \leq \|\boldsymbol{x}\|_p \leq n^{1/p}\|\boldsymbol{x}\|_\infty \leq n^{1/p}\|\boldsymbol{x}\|_q.$$

■

■ **Example 1.16** The uniform and integral metrics on $C^0([0,1],\mathbb{R})$ are not equivalent. Indeed, consider the sequence of functions $\{f_n\}$ in $C^0([0,1],\mathbb{R})$ defined by

$$f_n(x) = \begin{cases} 1-nx & \text{if } 0 \le x \le 1/n; \\ 0 & \text{if } 1/n < x \le 1. \end{cases} \tag{1.4}$$

Let us denote by 0_f the identically 0 function (i.e. $0_f(x) = 0$ for every $x \in [0,1]$). We have that

$$\|f_n - 0_f\|_{L^1([0,1])} = \int_0^1 |f_n(t)|\,\mathrm{d}t = \int_0^{1/n} (1-nt)\,\mathrm{d}t = \frac{1}{2n},$$

and so $\|f_n - 0_f\|_{L^1([0,1])} \to 0$, i.e. $\{f_n\}$ converges to the identically 0 function in $C^0([0,1],\mathbb{R})$ with the integral metric.

On the other hand, for each $n \in \mathbb{N}$, we have $\|f_n - 0_f\|_\infty = \sup_{x\in[0,1]} |f_n(x) - 0_f(x)| = 1$ and so $\{f_n\}$ does not converge to the identically 0 function in $C^0([0,1],\mathbb{R})$ with the uniform metric. By Corollary 1.1 and Lemma 1.5, we obtain the desired assertion. ■

Exercise 1.9 Let (X,d) be a metric space and let d' be the metric on X defined in Exercise 1.1 as $d'(x,y) = \frac{d(x,y)}{1+d(x,y)}$. Show that d and d' are not necessarily equivalent but nevertheless define the same open sets. Enter ■

We finally conclude this introductory chapter with the fundamental notion of continuity. The reader is certainly familiar with the notions of vector space, group and graph. Each of these objects essentially consist of a set together with a specific "structure"[2]. In order to understand the relationship between two vector spaces, one is interested in linear maps. Similarly, for groups and graphs, one studies group homomorphisms and graph homomorphisms. All these maps respect the "structure" of the objects in question.

In the case of metric spaces, we have that the property to be preserved is "closeness" and functions satisfying this property are called continuous. The classical ε,δ-definition of continuity in $\mathbb{R}$ can be easily adapted to the metric setting.

Notation 1.1. *Let $f\colon A \to B$ be a function between two sets A and B and let $A' \subset A$ and $B' \subset B$. We denote by $f(A')$ the image of A' under f, i.e. $f(A') = \{b \in B : f(a) = b$ for some $a \in A'\}$, and by $f^{-1}(B')$ the preimage of B' under f, i.e. $f^{-1}(B') = \{a \in A : f(a) \in B'\}$.*

Definition 1.10 — Continuous function. Let (X,d_X) and (Y,d_Y) be metric spaces and let $x_0 \in X$. The function $f\colon X \to Y$ is continuous at x_0 if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(x_0)) < \varepsilon$ for every $x \in X$ with $d_X(x,x_0) < \delta$.

In other words, f is continuous at x_0 if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $B_\delta(x_0) \subset f^{-1}(B_\varepsilon(f(x_0)))$.

The function f is continuous if it is continuous at each point of X.

We now provide a characterization of continuity (at a point) in terms of convergent sequences:

Theorem 1.1 Let (X,d_X) and (Y,d_Y) be metric spaces and let $x_0 \in X$. The following assertions are equivalent for $f\colon X \to Y$:

(i). f is continuous at x_0.

[2] We have seen that a metric space is nothing but a set together with a "structure" given by a metric.

(ii). For each neighbourhood N of $f(x_0)$, we have that $f^{-1}(N)$ is a neighbourhood of x_0.
(iii). For each sequence $\{x_n\}$ in X convergent to x_0, the sequence $\{f(x_n)\}$ converges to $f(x_0)$.

Proof. (i) $\Rightarrow$ (ii) Let N be a neighbourhood of $f(x_0)$. By Proposition 1.2, there exists $\varepsilon > 0$ such that $B_\varepsilon(f(x_0)) \subset N$. Moreover, by Definition 1.10, there exists $\delta > 0$ such that $B_\delta(x_0) \subset f^{-1}(B_\varepsilon(f(x_0))) \subset f^{-1}(N)$ and so $f^{-1}(N)$ is a neighbourhood of x_0.

(ii) $\Rightarrow$ (iii) Let $\{x_n\}$ be a sequence in X convergent to x_0 and let N be a neighbourhood of $f(x_0)$. By assumption, $f^{-1}(N)$ is a neighbourhood of x_0 and so, since $x_n \to x_0$, there exists $n_N \in \mathbb{N}$ such that $x_n \in f^{-1}(N)$ for every $n \geq n_N$. Therefore, $f(x_n) \in N$ for every $n \geq n_N$. Since N is an arbitrary neighbourhood of $f(x_0)$, we have $f(x_n) \to f(x_0)$.

(iii) $\Rightarrow$ (i) We proceed by contradiction. Suppose there exists $\varepsilon_0 > 0$ such that, for each $\delta > 0$, there exists $x_\delta \in X$ with $d_X(x_\delta, x_0) < \delta$ but $d_Y(f(x_\delta), f(x_0)) \geq \varepsilon_0$. We now define a new sequence $\{z_n\}$ in X by letting $z_n = x_{1/n}$. By the definition of x_δ, we have $d_X(z_n, x_0) < \frac{1}{n}$ and so z_n converges to x_0. On the other hand $d_Y(f(z_n), f(x_0)) \geq \varepsilon_0$ for each $n \in \mathbb{N}$ and so $\{f(z_n)\}$ does not converge to $f(x_0)$, a contradiction. ■

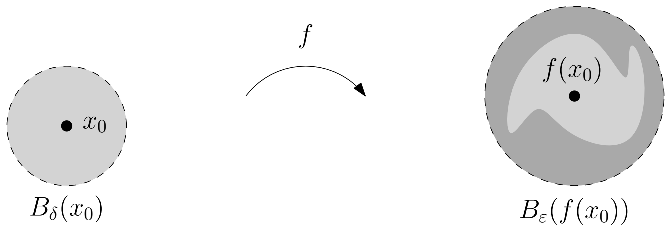


Figure 1.4: Continuous function at x_0.

We can finally provide a "topological" characterization of (global) continuity involving open sets:

Theorem 1.2 Let (X, d_X) and (Y, d_Y) be metric spaces. The following assertions are equivalent for $f\colon X \to Y$:

(i). f is continuous.
(ii). $f^{-1}(U)$ is open in X for each open set U of Y.
(iii). $f^{-1}(U)$ is closed in X for each closed set U of Y.

Proof. (i) $\Rightarrow$ (ii) Let U be an open set of Y. For each $x \in f^{-1}(U)$, we have that U is a neighbourhood of $f(x)$. Since f is continuous, Theorem 1.1 implies that $f^{-1}(U)$ is a neighbourhood of x, for each $x \in f^{-1}(U)$. Therefore, $f^{-1}(U)$ is a neighbourhood of each of its points and so Proposition 1.2 implies it is open.

(ii) $\Rightarrow$ (i) In view of Theorem 1.1, we show that for each $x_0 \in X$ and for each neighbourhood N of $f(x_0)$, the set $f^{-1}(N)$ is a neighbourhood of x_0. Therefore, let $x_0 \in X$ and N be a neighbourhood of $f(x_0)$. This means there exists an open set U in Y such that $f(x_0) \in U \subset N$. By (ii), we have that $f^{-1}(U)$ is open in X and $x_0 \in f^{-1}(U) \subset f^{-1}(N)$. Therefore, $f^{-1}(N)$ is a neighbourhood of x_0.

(ii) $\Leftrightarrow$ (iii) We just show one direction, as the other can just be obatined by changing the word closed with open. If U is a closed set of Y, then $Y \setminus U$ is open. Therefore, $f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$ is open and so $f^{-1}(U)$ is closed. ■

■ **Example 1.17** Every function from a discrete metric space to an arbitrary metric space is continuous. Indeed, in a discrete metric space every set is open. ■

An important class of continuous functions is given by the following:

■ **Example 1.18 — Lipschitz function.** Let (X,d_X) and (Y,d_Y) be metric spaces. A function $f\colon X\to Y$ is Lipschitz if there exists $L>0$ such that, for every $x,y\in X$, we have $d_Y(f(x),f(y))\le L\cdot d_X(x,y)$.

Lipschitz functions are continuous. Indeed, for each $x\in X$ and $\varepsilon>0$, we have that $f(B_{\varepsilon/L}(x))\subset B_\varepsilon(f(x))$. ■

Exercise 1.10 Consider the Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ with the Euclidean metric and a linear map $f\colon \mathbb{R}^n\to\mathbb{R}^m$. Show that f is Lipschitz and so continuous. Enter ■

Exercise 1.11 Let (X,d) be a metric space and $A\subset X$ non-empty. The distance of a point $x\in X$ from A is the quantity $d(x,A)=\inf_{z\in A} d(x,z)$.

Show that the function from (X,d) to $\mathbb{R}$ (with the Euclidean metric) defined by $x\mapsto d(x,A)$ is Lipschitz. Enter ■

Theorem 1.2 is extremely important, as it highlights the fact that in order to talk about continuity we don't need to have a metric structure: the only essential requirement is to know the open (or closed) sets. This suggests that constructions having an apparent "analytic nature" might carry on in a more general context, provided we can properly define the notion of open sets. In fact, in the next section, we will define a topological space as a set equipped with some distinguished subsets, the open sets, satisfying the properties (i) to (iii) in Proposition 1.1.

Exercise 1.12 Let $\mathbb{R}$ be equipped with the Euclidean metric. Find an example of a continuous function $f\colon\mathbb{R}\to\mathbb{R}$ and an open set $U\subset\mathbb{R}$ such that $f(U)$ is not open.

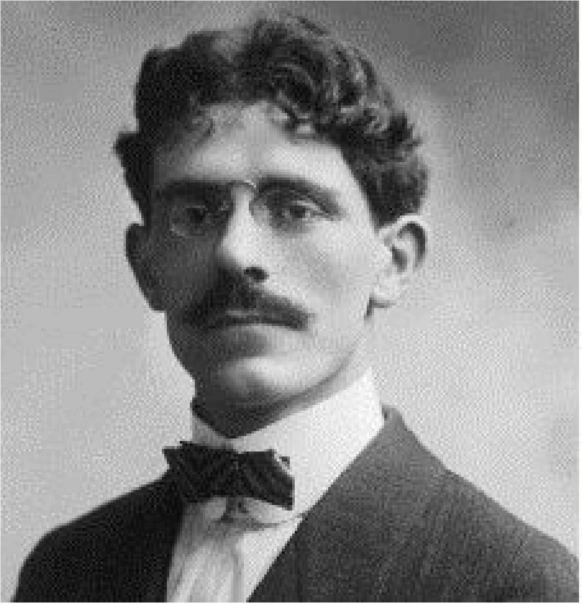
Similarly, show there exist a continuous function $f\colon\mathbb{R}\to\mathbb{R}$ and a closed set $V\subset\mathbb{R}$ such that $f(V)$ is not closed. Enter ■

Exercise 1.13 Show that the following set is open (in $\mathbb{R}^2$ with the Euclidean metric):

$$\left\{(x,y)\in\mathbb{R}^2 : \frac{x}{\sin(y)+x^2}>1\right\}.$$

■

Exercise 1.14 Give an example of a continuous bijection f of metric spaces such that f^{-1} is not continuous. Enter ■



2. Topological Spaces

We begin this section by defining topological spaces. Their study not only allows to generalize many familiar results but also helps understanding what are the essential properties that make certain theorems work.

Rather than by a metric (which is a quantitative notion), a topological space is described by giving a collection of "open sets" (a qualitative notion). These "open sets" should somehow behave in the same way as open sets in a metric space and so we use the properties in Proposition 1.1 as axioms:

Definition 2.1 — Topology and open sets. A topology on a set X is a family $\mathscr{T}$ of subsets of X satisfying the following properties:

(i). $\varnothing$ and X are in $\mathscr{T}$;
(ii). The union of the elements of any subfamily of $\mathscr{T}$ is in $\mathscr{T}$;
(iii). The intersection of the elements of any finite subfamily of $\mathscr{T}$ is in $\mathscr{T}$.

A topological space is a pair $(X,\mathscr{T})$, where X is a set and $\mathscr{T}$ is a topology on X. The sets in $\mathscr{T}$ are called open sets.

Notation 2.1. *The elements of X are usually called points and when no ambiguity may arise we refer to the topological space $(X,\mathscr{T})$ simply as X. If $U \in \mathscr{T}$, we say that U is open in X, or that U is an open set of X.*

By construction, every metric space is a topological space:

■ **Example 2.1 — Metric topology.** If (X,d) is a metric space, the family of open sets of (X,d) is a topology on X (Proposition 1.1), the metric topology.

We refer to the metric topology on $\mathbb{R}^n$ coming from the Euclidean metric as to the Euclidean topology and unless stated otherwise $\mathbb{R}^n$ will always be equipped with the Euclidean topology. ■

A set has many topologies:

■ **Example 2.2 — Discrete and indiscrete topology.** Given a set X, the family of all subsets of X is a topology, the discrete topology. The indiscrete topology is another trivial topology on X consisting of the family $\{\varnothing, X\}$. ■

■ **Example 2.3 — Cofinite topology.** Given a set X, the family of all subsets U of X such that $X \setminus U$ is either finite or X is a topology. This follows easily by De Morgan's laws. ■

■ **Example 2.4 — Cocountable topology.** Given a set X, the family of all subsets U of X such that $X \setminus U$ is either countable or X is a topology. ■

Exercise 2.1 — Excluded point topology. Show that, given a non-empty set X and a point $p \in X$, the family of all subsets of X not containing p together with the set X is a topology.
Enter ■

Note that Example 2.2 implies that not every topology comes from a metric. Indeed, consider a set X with more than one element. If the indiscrete topology on X coincides with a metric topology (i.e. they have the same open sets), then the complement of each one-element subset of X is open in the indiscrete topology, but this is clearly not the case as $|X| > 1$. The previous observation suggests the following:

Definition 2.2 — Metrizable space. A topological space $(X, \mathscr{T})$ is metrizable if there exists a metric on X such that the metric topology on X coincides with $\mathscr{T}$, i.e they have the same open sets.

Let us now provide another example of a topological space which is not metrizable:

■ **Example 2.5** Let $(X, \mathscr{T})$ be a topological space such that X is finite and $\mathscr{T}$ is not the discrete topology. We claim that $(X, \mathscr{T})$ is not metrizable. Clearly, it is enough to show that for every metrizable topological space $(X, \mathscr{T})$ such that X is finite, $\mathscr{T}$ is the discrete topology.

Indeed, since $(X, \mathscr{T})$ is metrizable, there exists a metric d on X such that $\mathscr{T}$ coincides with the metric topology (induced by d). Moreover, the finitness of X implies that $\{d(x,y) : x, y \in X\}$ is a finite set and let $\varepsilon = \min\{d(x,y) : x, y \in X\}$. But then, for each $x \in X$, we have that $B_\varepsilon(x) \subset \{x\}$. Therefore, $\{x\}$ is open and so so every subset of X is open. ■

As in the case of metrics, it is natural to compare two topologies on a certain set:

Definition 2.3 Let $\mathscr{T}$ and $\mathscr{T}'$ be two topologies on X. If $\mathscr{T}' \supset \mathscr{T}$, the topology $\mathscr{T}'$ is finer than $\mathscr{T}$ or $\mathscr{T}$ is coarser than $\mathscr{T}'$; if $\mathscr{T}'$ properly contains $\mathscr{T}$, the topology $\mathscr{T}'$ is strictly finer than $\mathscr{T}$ or $\mathscr{T}$ is strictly coarser than $\mathscr{T}'$. The topology $\mathscr{T}$ is comparable with $\mathscr{T}'$ if either $\mathscr{T}' \supset \mathscr{T}$ or $\mathscr{T} \supset \mathscr{T}'$.

Exercise 2.2 Are the cocountable and metric topology on $\mathbb{R}$ comparable? ■

In parallel with the case of metric spaces, we have the following:

Definition 2.4 — Closed set. Given a topological space $(X, \mathscr{T})$, a subset $U \subset X$ is closed if $X \setminus U \in \mathscr{T}$, i.e. $X \setminus U$ is open.

Proposition 2.1 For a topological space $(X, \mathscr{T})$, the following hold:

(i). $\varnothing$ and X are in $\mathscr{T}$;
(ii). Arbitrary intersections of closed sets are closed;
(iii). Finite unions of closed sets are closed.

Definition 2.5 — Neighbourhood. Let $(X, \mathscr{T})$ be a topological space and let $x \in X$. A subset N of X is a neighbourhood of x if there exists an open subset U of X such that $x \in U \subset N$.

Definition 2.6 — Interior, boundary, closure, derived. Given a subset S of a topological space X and a point $x \in X$, exactly one of the following three possibilities holds:

(i). There exists an open set U in X with $x \in U \subset S$.
(ii). There exists an open set U in X with $x \in U \subset X \setminus S$.
(iii). Every open set U with $x \in U$ has non-empty intersection with both S and $X \setminus S$.

If x satisfies (i), then it is an interior point of S and the set of all interior points of S is the interior of S, denoted by $\mathrm{int}(S)$.

If x satisfies (iii), then it is a boundary point of S and the set of all boundary points of S is the boundary of S, denoted by ∂S.

If x is either an interior point or a boundary point of S, then it is an adherent point of S and the set of all adherent points of S is the closure of S, denoted by $\overline{S}$.

If x belongs to the closure of $S \setminus \{x\}$, then it is a limit point of S and the set of all limit points of S is the derived of S, denoted by $D(S)$.

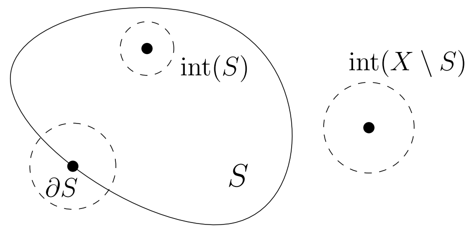


Figure 2.1: The possible "positions" of a point in X with respect to S.

Clearly, $X = \mathrm{int}(S) \cup \mathrm{int}(X \setminus S) \cup \partial S$, where the sets in the union are pairwise disjoint. Moreover, we have the following chain of inclusions:

$$\mathrm{int}(S) \subset S \subset \mathrm{int}(S) \cup \partial S = \overline{S}.$$

In other words, $x \in X$ is an interior point of S if and only if S is a neighbourhood of x. Note also that $x \in X$ is an adherent point of S if and only if every neighbourhood of x intersects S and x is a limit point if and only if every neighbourhood of x intersects S in a point distinct from x.

■ **Example 2.6** In $\mathbb{R}$ with the Euclidean topology, we have $\mathrm{int}(\mathbb{Q}) = \varnothing$, $\partial \mathbb{Q} = \mathbb{R}$ and so $\overline{\mathbb{Q}} = \mathbb{R}$. If S is any of the intervals $(a,b], [a,b), [a,b], (a,b)$, we have $\mathrm{int}(S) = (a,b)$, $\partial S = \{a,b\}$ and $\overline{S} = [a,b]$. If $S = \{1/n : n \in \mathbb{N}\}$, then $\mathrm{int}(S) = \varnothing$ and $0 \in \overline{S}$. ■

Proposition 2.2 Let X be a topological space and $S \subset X$. The following hold:

(i). $\mathrm{int}(S)$ is open;
(ii). $\overline{S}$ is closed;
(iii). S is open if and only if $S = \mathrm{int}(S)$;
(iv). S is closed if and only if $S = \overline{S}$;
(v). $\mathrm{int}(S)$ is the union of all open sets contained in S, i.e. it is the largest open set contained in S;
(vi). $\overline{S}$ is the intersection of all closed sets containing S, i.e. it is the smallest closed set containing S.
(vii). $\overline{S} = S \cup D(S)$.

Proof. (i) For each $x \in \mathrm{int}(S)$, there exists an open set U_x in X with $x \in U_x \subset S$. Moreover, $U_x \subset \mathrm{int}(S)$ and so $\mathrm{int}(S) = \bigcup_{x \in \mathrm{int}(S)} U_x$ is open.

(ii) Since X is the union of the pairwise disjoint subsets $\mathrm{int}(S)$, $\mathrm{int}(X \setminus S)$ and ∂S, we have that $\overline{S}$ is the complement of $\mathrm{int}(X \setminus S)$ (in X) and so it is closed by (i).

(iii) S is open if and only if S is a neighbourhood of each of its points (indeed, check that the proof given in Proposition 1.2 in the context of metric spaces holds unchanged in the context of topological spaces!). Moreover, S is a neighbourhood of each of its points if and only if $S \subset \mathrm{int}(S)$ and the conclusion follows.

(iv) S is closed if and only if $X \setminus S$ is open. Therefore, by (iii), S is closed if and only if $X \setminus S = \mathrm{int}(X \setminus S) = X \setminus \overline{S}$, from which the conclusion follows.

(v) Let U be an open set contained in S. For every $x \in U$, the set U is a neighbourhood of x and so $x \in \mathrm{int}(S)$. Similarly, we obtain the other inclusion.

(vi) It follows easily from the fact that $\overline{S}$ is the complement of $\mathrm{int}(X \setminus S)$ (in X) and (v).

(vii) Clearly, $D(S) \cap \mathrm{int}(X \setminus S) = \varnothing$ and so $S \cup D(S) \subset \overline{S}$. Consider now $x \in \overline{S}$. If $x \notin S$, then $x \in \partial S$ and so each neighbourhood of x intersects S in a point distinct from x. Therefore, $x \in D(S)$ and $\overline{S} \subset S \cup D(S)$. ■

■ **Example 2.7** In $\mathbb{R}$ with the cofinite topology the closed sets are the finite sets and $\mathbb{R}$. Therefore, we have $\overline{(a,b)} = \mathbb{R}$, $\mathrm{int}((a,b)) = \varnothing$ and $\partial(a,b) = \mathbb{R}$. ■

■ **Example 2.8** Consider $\mathbb{R}^2$ with the Euclidean topology and the subset $S = \{(x,y) \in \mathbb{R}^2 : y = 0\}$. We have that S is closed. Indeed, if $(x,y) \in \mathbb{R}^2 \setminus S$, then $y \neq 0$ and $B_{|y|}((x,y)) \subset \mathbb{R}^2 \setminus S$. Therefore, $\overline{S} = S$. Moreover, $\mathrm{int}(S) = \varnothing$. Indeed, if $(x,0) \in \mathrm{int}(S)$ then, for every $\varepsilon > 0$, the open ball $B_\varepsilon((x,0))$ contains the point $(x, \varepsilon/2) \notin S$. Finally, $\partial S = S$. ■

■ **Example 2.9** In a metric space X, the closure of the open ball $B_\varepsilon(x)$ is not necessarily the closed ball centered at x with radius ε.

Indeed, if X is a discrete metric space, then the closed ball centered at x with radius 1 is just X itself. Therefore, if $|X| > 1$, we have that $\overline{B_1(x)} = \overline{\{x\}} = \{x\} \subsetneq X$. ■

Exercise 2.3 Let A a subset of a topological space. Show that $\partial A = \varnothing$ if and only if A is both open and closed. ■

Exercise 2.4 Show that in a normed space the closure of the open ball centered at x with radius ε is the closed ball centered at x with radius ε. Enter ■

Exercise 2.5 Let A and B be subsets of a topological space. Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$ and $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$. Enter ■

Definition 2.7 — Dense subset. A subset $S \subset X$ of a topological space X is dense in X if $\overline{S} = X$.

A canonical example of a dense subset of $\mathbb{R}$ (with the Euclidean topology) is $\mathbb{Q}$.

■ **Example 2.10** Let $(X, \mathscr{T})$ be a topological space. X is the only dense subset of $(X, \mathscr{T})$ if and only if $\mathscr{T}$ is the discrete topology.

Indeed, if X is the only dense subset of $(X, \mathscr{T})$, then $\overline{X \setminus \{x\}} = X \setminus \{x\}$ for each $x \in X$ and so $\{x\}$ is open for each $x \in X$. The other implication follows from the fact that every subset of a discrete topological space is closed. ■

Exercise 2.6 — Dyadic rational numbers. The set D of dyadic rational numbers is defined as follows. $D = \bigcup_{n \geq 0} D_n$, where $D_0 = \{0,1\}$ and, for $n \geq 1$, $D_n = \{\frac{a}{2^n} : a \in \mathbb{N},\ a \text{ odd},\ 0 < a < 2^n\}$.

Show that D is dense in $[0,1]$ (with the Eucildean topology). Enter ■

Proposition 2.3 Let X be a topological space and $S \subset X$. The following are equivalent:

(i). S is dense in X, i.e. $\overline{S} = X$.
(ii). $\operatorname{int}(X \setminus S) = \varnothing$.
(iii). S intersects every non-empty open subset of X.

Proof. (i) $\Rightarrow$ (ii) It follows from the fact that X is the disjoint union of $\overline{S}$ and $\operatorname{int}(X \setminus S)$.

(ii) $\Rightarrow$ (iii) Suppose U is a non-empty open subset such that $U \cap S = \varnothing$ and let $x \in U$. Clearly, $x \in \operatorname{int}(X \setminus S)$, a contradiction.

(iii) $\Rightarrow$ (i) Obvious. ■

Definition 2.8 — Separable space. A topological space is separable if it contains a countable dense subset.

■ **Example 2.11** A typical example of a separable space is $\mathbb{R}^n$ with the Euclidean topology and the countable dense subset $\mathbb{Q}^n$. ■

■ **Example 2.12** Consider the metric space $B([0,1],\mathbb{R})$ of all bounded functions[1] from $[0,1]$ to $\mathbb{R}$ with the uniform metric. We claim it is not separable.

Indeed, let $f_a \in B([0,1],\mathbb{R})$ be given by

$$f_a(x) = \begin{cases} 0 & \text{if } x \neq a; \\ 1 & \text{if } x = a. \end{cases} \tag{2.1}$$

Clearly, the family $\{f_a\}_{a\in[0,1]}$ is uncountable. Observe now that the open balls $B_{1/4}(f_a)$ in $B([0,1],\mathbb{R})$ are pairwise disjoint. Indeed, if there exists $g \in B_{1/4}(f_{a_1}) \cap B_{1/4}(f_{a_2})$, for some $a_1 \neq a_2$, then

$$\sup_{x\in[0,1]} |f_{a_1}(x) - f_{a_2}(x)| \leq \sup_{x\in[0,1]} |f_{a_1}(x) - g(x)| + \sup_{x\in[0,1]} |g(x) - f_{a_2}(x)| < \frac{1}{2},$$

a contradiction. Suppose now S is a dense subset of $B([0,1],\mathbb{R})$. Since it intersects every open ball, taking $g_a \in S \cap B_{1/4}(f_a)$, we obtain an uncountable subset $\{g_a\}_{a\in[0,1]}$ of S and so S itself is uncountable.

On the other hand, it can be shown that $C^0([0,1],\mathbb{R})$ is indeed separable. ■

2.1 Convergence

We can finally introduce the notion of convergence in a topological space. It should come as no surprise that we adopt a definition in terms of neighbourhoods:

Definition 2.9 — Convergence. Let X be a topological space. A sequence $\{x_n\}$ in X converges to x if, for each neighbourhood N of x, there exists $n_N \in \mathbb{N}$ such that $x_n \in N$ for every $n \geq n_N$.

[1] Recall that $f\colon [0,1] \to \mathbb{R}$ is bounded if there exists $M > 0$ such that $|f(x)| \leq M$ for every $x \in [0,1]$.

Clearly, if a sequence converges in some topology on X it also converges in any coarser topology but not necessarily in a finer topology.

We have seen two desirable properties of convergence that hold in a metric space: uniqueness of limits and the fact that closed sets can be characterized in terms of convergent sequences. It turns out that none of them holds, in general, in a topological space.

Let us consider first uniqueness of limits:

■ **Example 2.13** Consider the excluded point topology (as defined in Exercise 2.1) on a set X (with $|X| > 1$) and let $p \in X$ be the excluded point. For $x \neq p$, the constant sequence $\{x_n\}$ with $x_n = x$ for each $n \in \mathbb{N}$ converges to both x and p. Indeed, the only neighbourhood of p is X, which clearly contains all the terms of the sequence. ■

Recalling the proof of uniqueness of limits in a metric space (Lemma 1.3), it should be noticed that the key property is the fact that any two distinct points have disjoint neighbourhoods. Loosely speaking, in a metric space we can find "enough" open sets. In order to rule out "pathological" examples like Example 2.13, we introduce the following:

Definition 2.10 — Hausdorff space. A topological space X is a Hausdorff space if, for each pair of distinct points $x, y \in X$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.

Clearly, every metric space is Hausdorff. In fact, most of the topological spaces that one encounters in Analysis and Geometry are Hausdorff and, in some sense, neighbourhoods in a Hausdorff space behave like those in a metric space. For example, the following result easily follows by rephrasing the proof of Lemma 1.3:

Lemma 2.1 Let X be a Hausdorff space, $\{x_n\}$ a sequence in X and $x, x' \in X$ such that $\{x_n\}$ converges to both x and x'. We have $x = x'$.

Note that the converse of Lemma 2.1 is in general not true, i.e. there exist topological spaces which are not Hausdorff even though each convergent sequence has a unique limit:

■ **Example 2.14** Let X be an uncountable set with the cocountable topology. X is not Hausdorff but uniqueness of limits holds.

Clearly, X is not Hausdorff, as every pair of non-empty open sets intersect. Suppose now there exists a sequence $\{x_n\}$ in X convergent to x and x' with $x \neq x'$. Consider the open set $U = X \setminus \{x_i : x_i \neq x\}$. Since $x \in U$, there exists $n_x \in \mathbb{N}$ such that $x_n \in U$ for every $n \geq n_x$. This implies that $x_n = x$ for every $n \geq n_x$. Similarly, there exists $n_{x'} \in \mathbb{N}$ such that $x_n = x'$ for every $n \geq n_{x'}$. Therefore, for every $n \geq \max\{n_x, n_{x'}\}$, we have $x = x_n = x'$, a contradiction. ■

Exercise 2.7 Consider $\mathbb{R}$ and the sequence $\{x_n\}$ with $x_n = 1/n$. Study the convergence of $\{x_n\}$ in the Euclidean topology, in the indiscrete topology, in the discrete topology and in the cofinite topology. ■

Another similarity between (neighbourhoods in) metric spaces and Hausdorff spaces is given by the following:

Lemma 2.2 Every finite point set in a Hausdorff space X is closed.

Proof. Since finite unions of closed sets are closed, it is enough to show that $\{x_0\}$ is closed in X, for every $x_0 \in X$. Indeed, if x is a point of X different from x_0, there exist disjoint open sets U and V containing x and x_0, respectively. Since U does not intersect $\{x_0\}$, the point x does not belong to the closure of $\{x_0\}$ and so $\overline{\{x_0\}} = \{x_0\}$. Therefore, $\{x_0\}$ is closed. ■

Exercise 2.8 Show that the only topology on a finite Hausdorff topological space is the discrete topology. ■

Consider now the characterization of closed sets in terms of convergent sequences (Lemma 1.4). In a general topological space, it does not hold:

■ **Example 2.15** Let X be an uncountable set with the cocountable topology. We claim that every subset of X contains the limit of each of its convergent sequences.

Indeed, let $S \subset X$ and let $\{x_n\}$ be a sequence in S convergent to x. We have that the set $X \setminus \{x_1, x_2, \dots\}$ is open. Since $\{x_n\}$ converges to x, the point x does not belong to $X \setminus \{x_1, x_2, \dots\}$ (for otherwise $X \setminus \{x_1, x_2, \dots\}$ is a neighbourhood of x) and so it must be $x \in \{x_1, x_2, \dots\} \subset S$.

Note that a proper uncountable subset of X is not closed and so Lemma 1.4 does not hold for an arbitrary topological space. ■

A careful look at the proof of Lemma 1.4 shows in fact that one implication carry on in the context of topological spaces (with the same proof!):

Lemma 2.3 Let X be a topological space and $U \subset X$ a closed subset. If $\{x_n\}$ is a sequence in U convergent to x, then $x \in U$.

The key property used in the converse implication was that, for every point x in a metric space, we can find countably many neighbourhoods $N_1, N_2, \dots$ of x such that every neighbourhood of x contains at least one of the N_i's (in the proof we took $N_i = B_{1/i}(x)$). We have that, in some sense, if a topological space does not satisfy this property, then there are "too many" neighbourhoods containing a point x to be able to describe closed sets in terms of sequences (which are by definition countable). Let us formalize the property above:

Definition 2.11 — First axiom of countability. A topological space X satisfies the first axiom of countability, or is first countable, if for each $x \in X$ there exists a sequence $\{N_n\}$ of neighbourhoods of x such that each neighbourhood of x contains at least one of the N_i's.

Clearly, every metric space is first countable. Another example is the following:

■ **Example 2.16** Every space X with the excluded point topology is first countable. Indeed, let $p \in X$ be the excluded point. For each $x \in X$, we define a neighbourhood N of x as follows:

$$N = \begin{cases} X & \text{if } x = p; \\ \{x\} & \text{otherwise.} \end{cases} \tag{2.2}$$

Clearly, every neighbourhood of x contains N. ■

The following result strengthens Lemma 1.4 and provides a characterization of adherent points in a first countable space.

Proposition 2.4 Let S be a subset of a topological space X. If a sequence $\{x_n\}$ in S converges to $x \in X$, then $x \in \overline{S}$. The converse holds if X is first countable.

Proof. Since $\{x_n\}$ converges to x, every neighbourhood of x contains a point of S and so $x \in \overline{S}$.

For the converse, suppose X is first countable and let $x \in \overline{S}$. We show there exists a sequence $\{x_n\}$ in S convergent to x. By the assumption on X, there exists a sequence $\{N_n\}$ of neighbourhoods of x such that each neighbourhood of x contains at least one of the N_i's. For each $n \in \mathbb{N}$, consider the neighbourhood $B_n = N_1 \cap \cdots \cap N_n$ of x and let x_n be a point in $B_n \cap S$ (this set is non-empty as $x \in \overline{S}$). We claim that the sequence $\{x_n\}$ converges to x. Indeed, every neighbourhood N of x contains N_i, for some i, and so we have that $x_n \in B_n \subset B_i \subset N_i \subset N$ for every $n \geq i$. ■

■ **Example 2.17** An uncountable set X with the cocountable topology does not satisfy the first axiom of countability.

Indeed, let S be a countable subset of X and let $x \in \overline{S}$. If X is first countable, Proposition 2.4 implies that x is the limit of a sequence in S. Therefore, by Example 2.15, $x \in S$ and so S is closed, a contradiction. ■

Exercise 2.9 Let X be a first countable topological space and $S \subset X$. Show that the point $x \in X$ is a limit point of S if and only if there exists a sequence $\{x_n\}$ in S taking distinct values and converging to x. ■

We now characterize limit points in a Hausdorff space:

Proposition 2.5 Let X be a Hausdorff space and $S \subset X$. The point $x \in X$ is a limit point of S if and only if every neighbourhood of x contains infinitely many points of S.

Proof. If every neighbourhood of x contains infinitely many points of S, then it clearly contains a point distinct from x and so x is a limit point of S.

Conversely, we show that if $x \in X$ has a neighbourhood containing only finitely many points of S, then x is not a limit point of S. Therefore, suppose N is such a neighbourhood and let $S' = \{s_1, \dots, s_n\}$ be the finitely many points distinct from x in $N \cap S$. Since X is Hausdorff, S' is closed (Lemma 2.2) and so $X \setminus S'$ is an open neighbourhood of x which does not intersect S in a point distinct from x. Therefore, x is not a limit point of S. ■

The first axiom of countability provides a sufficient condition for the converse of Lemma 2.1. In other words, in a first countable space, the Hausdorff condition is exactly what guarantees uniqueness of limits:

Lemma 2.4 If X is a first countable topological space such that every convergent sequence in X has a unique limit, then X is Hausdorff.

Proof. Suppose, to the contrary, X is not Hausdorff. This implies there exist two distinct points x and y such that, for every open sets U and V with $x \in U$ and $y \in V$, we have $U \cap V \neq \varnothing$. Since X is first countable, there exist sequences of neighbourhoods $\{U_n\}$ and $\{V_n\}$ of x and y, respectively, such that each neighbourhood of x contains one of the U_i's and each neighbourhood of y contains one of the V_i's. As in the proof of Proposition 2.4, for each $i \in \mathbb{N}$, consider the neighbourhoods $N_i = U_1 \cap \cdots \cap U_i$ of x and $M_i = V_1 \cap \cdots \cap V_i$ of y. For each $i \in \mathbb{N}$, choose $z_i \in N_i \cap M_i$. Clearly, the sequence $\{z_n\}$ converges to both x and y, a contradiction. ■

2.2 Bases

We have seen that in a metric space a subset is open if and only if it is a union of open balls. This motivates the following:

Definition 2.12 A family $\mathscr{B}$ of open subsets of a topological space X is a base for the topology of X if every open subset of X is a union of sets in $\mathscr{B}$.

■ **Example 2.18** If (X,d) is a metric space, then $\mathscr{B} = \{B_r(x) : x \in X, r > 0\}$ and $\mathscr{B}' = \{B_{1/n}(x) : x \in X, n \in \mathbb{N}\}$ are bases. ■

Exercise 2.10 Let X be a topological space. A subset $S \subset X$ is dense if and only if there exists a base $\mathscr{B}$ for the topology of X such that S intersects every non-empty open subset in $\mathscr{B}$. Enter ■

Note that the notion of base for a topology is conceptually different from the well-known basis in linear algebra, as a given open subset can be a union of sets in a base in many different ways.

Lemma 2.5 A family $\mathscr{B}$ of open subsets of a topological space X is a base for the topology of X if and only if for each $x \in X$ and each neighbourhood N of x, there exists $U \in \mathscr{B}$ such that $x \in U \subset N$.

Proof. Let $\mathscr{B}$ be a base, $x \in X$ and N a neighbourhood of x. There exists an open subset V such that $x \in V \subset N$ and V is a union of open sets in $\mathscr{B}$. Therefore, there exists $U \in \mathscr{B}$ such that $x \in U \subset N$.

Conversely, suppose that for each $x \in X$ and each neighbourhood N of x, there exists $U \in \mathscr{B}$ with $x \in U \subset N$. Let V be an open subset of X. By assumption, for each $x \in V$, there exists $V_x \in \mathscr{B}$ such that $x \in V_x \subset V$. Therefore, $V = \bigcup_{x \in V} V_x$ and so $\mathscr{B}$ is a base. ■

A topological space can have many different bases, as the following examples show. Moreover, these bases may have different cardinalities.

■ **Example 2.19** Consider $\mathbb{R}^n$ with the Euclidean topology. We claim that $\mathscr{B} = \{B_{1/n}(q) : q \in \mathbb{Q}^n, n \in \mathbb{N}\}$ is a base for the Euclidean topology.

Indeed, let $x \in X$ and N be a neighbourhood of x. By definition, there exist an open subset U such that $x \in U \subset N$ and $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U \subset N$. Moreover, we can find $n_0 \in \mathbb{N}$ with $1/n_0 < \varepsilon/2$ (by the Archimedean property of $\mathbb{R}$) and $q \in \mathbb{Q}^n$ such that $d(x,q) < 1/n_0$ (as $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$). This implies that $x \in B_{1/n_0}(q)$. We now show that $B_{1/n_0}(q) \subset B_\varepsilon(x)$. Indeed, if $y \in B_{1/n_0}(q)$, then $d(y,q) < 1/n_0$ and so $d(y,x) \leq d(y,q) + d(q,x) < 1/n_0 + 1/n_0 = 2/n_0 < \varepsilon$. ■

■ **Example 2.20** Consider $\mathbb{R}^2$ with the Euclidean topology and, for $\varepsilon > 0$ and $x \in \mathbb{R}^2$, the subset

$$Q_\varepsilon(x) = \left\{ (y_1, y_2) \in \mathbb{R}^2 : |y_1 - x_1| < \frac{\varepsilon}{2}, |y_2 - x_2| < \frac{\varepsilon}{2} \right\}.$$

We claim that $\mathscr{B} = \{Q_\varepsilon(x) : x \in \mathbb{R}^2, \varepsilon > 0\}$ is a base for the Euclidean topology.

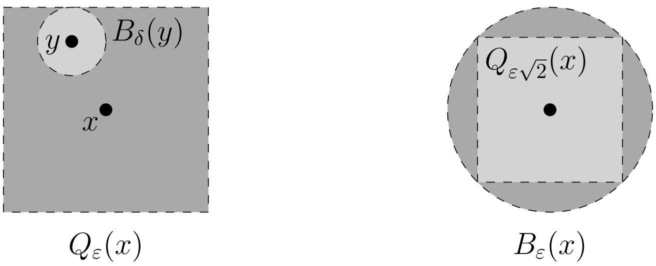


Figure 2.2:

Let us first show that $Q_\varepsilon(x)$ is open, for each $\varepsilon > 0$ and $x \in \mathbb{R}^2$. It is enough to show that if $y \in Q_\varepsilon(x)$ and $\delta = \min\{\varepsilon/2 - |y_1 - x_1|, \varepsilon/2 - |y_2 - x_2|\}$, then $B_\delta(y) \subset Q_\varepsilon(x)$. Indeed, if $z \in B_\delta(y)$ then, for each $i \in \{1,2\}$, we have that $|z_i - x_i| \leq |z_i - y_i| + |y_i - x_i| < \delta + |y_i - x_i| \leq \varepsilon/2$.

Let us finally show that, for each $x \in \mathbb{R}^2$ and each neighbourhood N of x, there exists $U \in \mathscr{B}$ such that $x \in U \subset N$. Clearly, it is enough to show that, for each $x \in \mathbb{R}^2$ and $\varepsilon > 0$, the open ball $B_\varepsilon(x)$ contains $Q_r(x)$ for some $r > 0$. But then just take $r = \varepsilon\sqrt{2}$. Indeed, if $y \in Q_r(x)$, then $d(y,x) = \sqrt{|y_1 - x_1|^2 + |y_2 - x_2|^2} < \sqrt{r^2/2} = \varepsilon$.

This example can be obviously generalized to higher dimensions. ■

The following exercise gives a handy criterion to check the relation between two topologies when we have knowledge of the corresponding bases:

Exercise 2.11 Let $\mathscr{B}$ and $\mathscr{B}'$ be bases for the topologies $\mathscr{T}$ and $\mathscr{T}'$, respectively, on a set X. We have that $\mathscr{T}' \supset \mathscr{T}$ if and only if, for every $x \in X$ and every $B \in \mathscr{B}$ containing x, there exists $B' \in \mathscr{B}'$ such that $x \in B' \subset B$. Enter ■

We have seen how a base can be determined starting from a topological space X. It is sometimes useful to proceed the other way round. Namely, given a set X and a family $\mathscr{B}$, under what conditions is $\mathscr{B}$ a base for a topology on X?

Proposition 2.6 A family $\mathscr{B}$ of subsets of X is a base for a topology of X if and only if the following hold:

(i). Each $x \in X$ is contained in at least one set in $\mathscr{B}$;
(ii). If $U, V \in \mathscr{B}$ and $x \in U \cap V$, then there exists $W \in \mathscr{B}$ such that $x \in W \subset U \cap V$.

Proof. If $\mathscr{B}$ is a base, then (i) and (ii) follow from the fact that X and $U \cap V$ are open.

Conversely, suppose that the family $\mathscr{B}$ satisfies (i) and (ii). Let $\mathscr{T}$ be the family of all subsets of X that are unions of sets in $\mathscr{B}$. We show that $\mathscr{T}$ is a topology. Clearly, $\varnothing \in \mathscr{T}$, $X \in \mathscr{T}$ by (i) and every union of sets in $\mathscr{T}$ is in $\mathscr{T}$. It remains to show that finite intersections of sets in $\mathscr{T}$ are in $\mathscr{T}$ and it is clearly enough to show this for intersections of two sets.

Therefore, let $U, V \in \mathscr{T}$ and $x \in U \cap V$. Since U and V are unions of sets in $\mathscr{B}$, there exist $U_0, V_0 \in \mathscr{B}$ such that $x \in U_0 \subset U$ and $x \in V_0 \subset V$ and so $x \in U_0 \cap V_0$. By (ii), there exists $W_x \in \mathscr{B}$ such that $x \in W_x \subset U_0 \cap V_0 \subset U \cap V$. Therefore, $U \cap V = \bigcup_{x \in U \cap V} W_x$ and so $U \cap V \in \mathscr{T}$. ■

The reader should notice that, given a family $\mathscr{B}$ of subsets of X satisfying the conditions in Proposition 2.6, there is a unique topology $\mathscr{T}$ of X for which $\mathscr{B}$ is a base. We say that $\mathscr{T}$ is the **topology generated** by $\mathscr{B}$.

Before showing an example let us recall the following:

Definition 2.13 — Ordered set. Given a set X, a linear order on X is a relation $<$ satisfying the following properties:

1. (Comparability) For every x and y in X such that $x \neq y$, either $x < y$ or $y < x$.
2. (Non-reflexivity) There is no $x \in X$ such that $x < x$.
3. (Transitivity) If $x < y$ and $y < z$, then $x < z$.

The pair $(X, <)$ is an ordered set.

Clearly, the usual order relation on $\mathbb{R}$ is a linear order. Given a linear order on X, we use the notation $x \leq y$ for the statement "$x < y$ or $x = y$". The conditions above imply that, for every $x, y \in X$, exactly one of the three relations $x < y$, $y < x$, $x = y$ holds.

Let $a, b \in X$ such that $a < b$. The *open interval* (a,b) is the set $\{x \in X : a < x < b\}$. The *closed interval* $[a,b]$ is the set $\{x \in X : a \leq x \leq b\}$. The *half-open intervals* $[a,b)$ and $(a,b]$ are the sets $\{x \in X : a \leq x < b\}$ and $\{x \in X : a < x \leq b\}$, respectively. Finally, the *open rays* $(a, +\infty)$ and $(-\infty, a)$ are the sets $\{x \in X : a < x\}$ and $\{x \in X : x < a\}$, respectively.

■ **Example 2.21 — Order topology.** Let $(X, <)$ be an ordered set. The family $\mathscr{B}$ of all open intervals, open rays and X itself is a base for a topology on X. Trivially, each $x \in X$ is contained in the open set X. Moreover, an easy case checking shows that condition (ii) in Proposition 2.6 is satisfied.

Note that the Euclidean topology and the order topology on $\mathbb{R}$ coincide. Indeed, if a set U is open in the Euclidean topology, then it is a union of open balls and each open ball is an open interval in the sense described above. Conversely, all open intervals and open rays are clearly open sets in the Euclidean topology. ▪

▪ **Example 2.22 — Sorgenfrey line.** Consider $\mathbb{R}$ and the family $\mathscr{B}$ of all half-open intervals of the form $[a,b) = \{x \in \mathbb{R} : a \leq x < b\}$. Proposition 2.6 immediately implies that $\mathscr{B}$ is a base for a topology on $\mathbb{R}$ and the topological space consisting of $\mathbb{R}$ and the topology generated by $\mathscr{B}$ is called the Sorgenfrey line, denoted by $\mathbb{R}_\ell$.

Note that the topology generated by $\mathscr{B}$ is strictly finer than the Euclidean topology. Indeed, we have $(a,b) = \bigcup_{a<c<b}[c,b)$ and every open set in the Euclidean topology is union of open intervals. Moreover, the open subset $[a,b)$ in the Sorgenfrey line is not open in $\mathbb{R}$ with the Euclidean topology.

In particular, $\mathbb{R}_\ell$ is Hausdorff. On the other hand, we will see that it is not metrizable. ▪

A natural question arises: How small (in terms of cardinality) can a base be? Clearly, $\mathscr{T}$ admits a finite base if and only if $\mathscr{T}$ is a finite family.

Definition 2.14 — Second axiom of countability. A topological space X satisfies the second axiom of countability, or is second countable, if there exists a countable base for the topology of X.

Second countability, similarly to first countability, limits the number of open sets in a topological space: if X is second countable, the cardinality of the topology is at most that of the power set of $\mathbb{N}$.

▪ **Example 2.23** $\mathbb{R}^n$ with the Euclidean topology is second countable by Example 2.19. ▪

Note that every second countable space is first countable. Indeed, if $\mathscr{B}$ is a countable base for the topology of X then, for each $x \in X$, we can build a sequence of neighbourhoods of x by taking those sets in $\mathscr{B}$ containing x.

On the other hand, the second axiom of countability is in fact much stronger than the first and there exist metric spaces which are not second countable:

▪ **Example 2.24** Consider an uncountable set X with the discrete metric. It is clearly first countable but not second countable. ▪

Lemma 2.6 If X is a second countable topological space, then it is separable.

Proof. Let $\mathscr{B}$ be a countable base for the topology of X. For each non-empty open set B in $\mathscr{B}$, choose a point x_B and let $S = \{x_B : B \in \mathscr{B}\}$. Clearly, S is countable. Moreover, S is dense by Exercise 2.10, as it intersects every non-empty subset in a base. ■

The converse of Lemma 2.6 does not hold in general:

▪ **Example 2.25** Consider the Sorgenfrey line $\mathbb{R}_\ell$. By Exercise 2.10, $\mathbb{Q}$ is a dense subset in $\mathbb{R}_\ell$ and so the Sorgenfrey line is separable. On the other hand, we claim it is not second countable. Indeed, suppose $\mathscr{B}$ is a countable base. By Lemma 2.5, for each $x \in \mathbb{R}_\ell$, there exists $B_x \in \mathscr{B}$ such that $x \in B_x \subset [x, x+1)$. But since the map $x \mapsto B_x$ is injective, we obtain a contradiction.

However, $\mathbb{R}_\ell$ is first countable: just consider the sequence $\{[x, x+1/n)\}$ of neighbourhoods of $x \in \mathbb{R}_\ell$. ▪

A closer look at Example 2.19 suggests that in a metrizable space the converse indeed holds:

Lemma 2.7 If X is a separable and metrizable topological space, then X is second countable.

Proof. We proceed as in Example 2.19. Let S be a countable dense subset in X and let $\mathscr{B} = \{B_q(x) : x \in S,\ q \in \mathbb{Q},\ q > 0\}$. Since $\mathscr{B}$ is clearly countable, it remains to show it is a base for the topology of X and so we rely on Lemma 2.5. It is then enough to show that, for each $x \in X$ and $\varepsilon > 0$, there exist $y \in S$ and a rational $q > 0$ such that $x \in B_q(y) \subset B_\varepsilon(x)$. Since S is dense, there exists $y \in B_{\varepsilon/3}(x) \cap S$. But then, choosing $q_0 \in (\varepsilon/3, 2\varepsilon/3) \cap \mathbb{Q}$, we have $x \in B_{q_0}(y) \subset B_\varepsilon(x)$. ■

Lemma 2.7 shows that the Sorgenfrey line is not metrizable.

2.3 Continuous functions

We can finally introduce the notion of continuity in its full generality:

Definition 2.15 Let X and Y be topological spaces and $x \in X$. A function $f\colon X \to Y$ is continuous at x if, for each neighbourhood N of $f(x)$, we have that $f^{-1}(N)$ is a neighbourhood of x. The function f is continuous if it is continuous at each $x \in X$.

Proposition 2.7 Let X and Y be topological spaces and $f\colon X \to Y$. Moreover, let $\mathscr{B}$ be a base for Y. The following are equivalent:

(i). f is continuous.
(ii). $f^{-1}(U)$ is open in X for each open set U of Y.
(iii). $f^{-1}(U)$ is closed in X for each closed set U of Y.
(iv). $f^{-1}(U)$ is open in X for each $U \in \mathscr{B}$.

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) The proof is exactly the same as in Theorem 1.2.

(ii) $\Rightarrow$ (iv) Obvious.

(iv) $\Rightarrow$ (ii) If U is an open subset of Y, then $U = \bigcup B_\alpha$, where each B_α belongs to $\mathscr{B}$. Therefore, $f^{-1}(U) = \bigcup f^{-1}(B_\alpha)$ is a union of open sets. ■

■ **Example 2.26** Consider $\mathbb{R}$ with the Euclidean topology and the Sorgenfrey line $\mathbb{R}_\ell$. The identity function $f\colon \mathbb{R} \to \mathbb{R}_\ell$ (i.e. $f(x) = x$, for each $x \in \mathbb{R}$) is not continuous. Indeed, $f^{-1}([a,b)) = [a,b)$ is not open in the Euclidean topology.

On the other hand, the identity function $g\colon \mathbb{R}_\ell \to \mathbb{R}$ is clearly continuous. ■

Exercise 2.12 — Subbase. A subbase for a topological space X is a family $\mathscr{S}$ of open sets such that the family of all finite intersections of sets in $\mathscr{S}$ is a base for Y.

Let X and Y be topological spaces and $\mathscr{S}$ a subbase for Y. A function $f\colon X \to Y$ is continuous if and only if $f^{-1}(S)$ is open in X for each $S \in \mathscr{S}$. Enter ■

Proposition 2.8 Let X, Y and Z be topological spaces.

(i). If $f\colon X \to Y$ is a constant function, then f is continuous;
(ii). If $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous, then $g \circ f\colon X \to Z$ is continuous.

Proof. (i) There exists $y_0 \in Y$ such that $f(x) = y_0$ for each $x \in X$. Given an open subset V in Y, $f^{-1}(V)$ is either X or $\varnothing$, depending on whether V contains y_0 or not. In either case, the preimage is open.

(ii) If U is open in Z, then $g^{-1}(U)$ is open in Y and $f^{-1}(g^{-1}(U))$ is open in X. But clearly $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$. ■

Theorem 2.1 Let X and Y be topological spaces. If $f\colon X \to Y$ is a function continuous at x_0 then, for each sequence $\{x_n\}$ in X convergent to x_0, the sequence $\{f(x_n)\}$ converges to $f(x_0)$. The converse holds if X is first countable.

Proof. The proof of sufficiency is exactly the same as the one in Theorem 1.1!

Therefore, suppose X is first countable. We proceed by contradiction. Let N be a neighbourhood of $f(x_0)$ and suppose that $f^{-1}(N)$ is not a neighbourhood of x_0. Let $\{N_n\}$ be a sequence of neighbourhoods of x_0 as in the definition of first countability and let $B_n = N_1 \cap \cdots \cap N_n$. Since $f^{-1}(N)$ is not a neighbourhood of x_0, we have that $X_n = (X \setminus f^{-1}(N)) \cap B_n$ is non-empty for each $n \in \mathbb{N}$ and so we construct a sequence $\{x_n\}$ in X with $x_n \in X_n$ for each $n \in \mathbb{N}$. Clearly, $\{x_n\}$ converges to x_0 (see the proof of Proposition 2.4) and so $\{f(x_n)\}$ converges to $f(x_0)$. On the other hand, for each $n \in \mathbb{N}$, $x_n \notin f^{-1}(N)$ and so $f(x_n) \notin N$, contradicting the fact that $f(x_n) \to f(x_0)$. ■

The following lemma generalizes the well-known fact that continuous real-valued functions agreeing at rational points agree everywhere.

Lemma 2.8 Let X and Y be two topological spaces with Y Hausdorff. If $f\colon X \to Y$ and $g\colon X \to Y$ are two continuous functions such that $f(x) = g(x)$ for each $x \in D$, where D is dense in X, then $f(x) = g(x)$ for each $x \in X$.

Proof. Suppose $f(x) \neq g(x)$ for some $x \in X$. Since Y is Hausdorff, there exist disjoint open sets U and V of Y such that $f(x) \in U$ and $g(x) \in V$. Moreover, x belongs to the open subset $f^{-1}(U) \cap g^{-1}(V)$ of X and so, since D is dense in X, there exists $d \in D \cap f^{-1}(U) \cap g^{-1}(V)$. But then $f(d) = g(d) \in U \cap V$, a contradiction. ■

Exercise 2.13 Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x)f(y) = f(x+y)$ for every $x, y \in \mathbb{R}$. Show that $f(x) = e^{cx}$ for some constant c. ■

Definition 2.16 — Homeomorphism. Let X and Y be topological spaces and $f\colon X \to Y$ be a bijection. The function f is a homeomorphism if both f and its inverse f^{-1} are continuous.

If $f\colon X \to Y$ is a homeomorphism, X and Y are homeomorphic and we write $X \cong Y$.

Suppose that $f\colon X \to Y$ is a homeomorphism. In particular, for each open set $U \subset Y$, its preimage $f^{-1}(U)$ is open in X[2]. On the other hand, let $V \subset X$ be open. Since $f^{-1}\colon Y \to X$ is continuous, $f(V)$ is open in Y. In other words, a homeomorphism gives a bijection not only between the points of X and Y, but also between the open sets of the two topologies. This means that X and Y are somehow the same topologically.

R A bijective continuous function need not be a homeomorphism (see Example 2.26).

To visualize Definition 2.16, the space X can be thought of as made of some elastic material and the space Y is homeomorphic to X if it can be obtained from X by changing its shape without tearing the material (by stretching, bending, ...).

[2]Note that the two possible meanings of $f^{-1}(U)$ coincide: the preimage is the same as the image of the inverse function.

We close this section by introducing the notion of a topological property. A property of topological spaces is a **topological property** if it is invariant under homeomorphisms, i.e. whenever a space X has the property, then every space homeomorphic to X has the property as well.

Exercise 2.14 Show that being first countable, second countable or Hausdorff are topological properties. Enter ■

2.4 Subspace topology

Given a topology $\mathscr{T}$ on a set X and a subset $S \subset X$, there is a natural way to define a topology $\mathscr{T}_S$ on S. Namely, just define $\mathscr{T}_S = \{U \cap S : U \in \mathscr{T}\}$:

Definition 2.17 — Subspace topology. Let X be a topological space with topology $\mathscr{T}$. If $S \subset X$, the family $\mathscr{T}_S = \{U \cap S : U \in \mathscr{T}\}$ is a topology on S, called the subspace topology. The set S together with the topology $\mathscr{T}_S$ is a subspace of X.

We leave as an easy exercise to check that $\mathscr{T}_S$ satisfies the three properties in Definition 2.1. Note that an open subset of the subspace S (i.e. a set in $\mathscr{T}_S$) need not be open in the space X: for example, $S \in \mathscr{T}_S$ but it may not belong to $\mathscr{T}$. Therefore, it should be stressed that openness and closedness are not properties of a set by itself, but rather of a set in relation to a certain topology. If S is a subspace of X, we say that a set U is open in S if it belongs to the topology of S, while U is open in X if it belongs to the topology of X. A similar vocabulary will be used for closed sets.

■ **Example 2.27** Consider $X = \mathbb{R}$ with the Euclidean topology and $S = [0,1]$ with the subspace topology. The sets $(0,1]$ and $[0,1)$ are open in S but not in X. The set $(0,1)$ is open both in X and S. The set $[0,1]$ is not open in X but it is open in S. ■

■ **Example 2.28** Consider $X = \mathbb{R}$ with the Euclidean topology and $S = \mathbb{Z}$ with the subspace topology. The subspace topology on S coincides with the discrete topology. Indeed, for every $n \in \mathbb{Z}$, we have that $\{n\} = \mathbb{Z} \cap (n - \frac{1}{2}, n + \frac{1}{2})$.

On the other hand, if $S = \mathbb{Q}$ the situation is different. Indeed, let V be an open subset of S containing a given rational number q. We have that $V = U \cap S$, for some open subset U of $\mathbb{R}$. Since U contains an interval $(q - \varepsilon, q + \varepsilon)$, for some $\varepsilon > 0$, the open subset V of S contains rational numbers distinct from q (as $\mathbb{Q}$ is dense in $\mathbb{R}$). ■

Exercise 2.15 Let S be a subspace of X. Show that a set C is closed in S if and only if it is the intersection of a closed set of X with S. Enter ■

As one would expect (and desire) the metric topology behaves well with respect to subspaces:

Lemma 2.9 The metric topology on a subset S of a metric space X coincides with the subspace topology.

Proof. Note that the intersection of an open ball $B_\varepsilon(x)$ in X with S consists of those points in S with distance less than ε from x and so it is an open ball in S. But since every open set in X is a union of open balls and $S \cap (\bigcup_i B_{\varepsilon_i}(x_i)) = \bigcup_i (S \cap B_{\varepsilon_i}(x_i))$, the two topologies have indeed the same open sets. ■

Exercise 2.16 Let X be a topological space and $S \subset X$. If $\mathscr{B}$ is a base for the topology of X, then $\mathscr{B}_S = \{B \cap S : B \in \mathscr{B}\}$ is a base for the subspace topology. Enter ■

■ **Example 2.29** For each $a < b$, the subspaces $[a,b]$ and $[0,1]$ of $\mathbb{R}$ are homeomorphic.

Indeed, the function $f\colon [a,b] \to [0,1]$ given by $f(x) = (x-a)/(b-a)$ is a continuous function with continuous inverse $f^{-1}\colon [0,1] \to [a,b]$ given by $f^{-1}(y) = y(b-a)+a$. ■

■ **Example 2.30** For each $a < b$, the subspace (a,b) of $\mathbb{R}$ is homeomorphic to $\mathbb{R}$.

Observe first that $(-\pi/2, \pi/2)$ is homeomorphic to $\mathbb{R}$ since $f\colon (-\pi/2,\pi/2) \to \mathbb{R}$ with $f(x) = \tan x$ is clearly a homeomorphism. Moreover, as in Example 2.29, it is easy to show that $(-\pi/2,\pi/2) \cong (a,b)$, for each $a < b$. ■

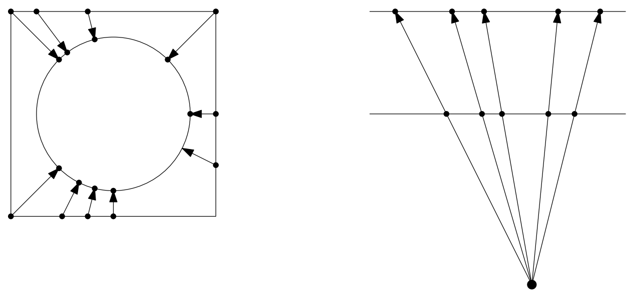


Figure 2.3: Homeomorphisms between the "empty square" and the "empty circle" (as subspaces in $\mathbb{R}^2$) and between the open intervals $(0,1)$ and (a,b).

In Analysis, we often encounter continuous functions that are defined differently on different intervals. The following result gives us a way to paste them together into a continuous function:

Lemma 2.10 — Pasting Lemma. Let X and Y be topological spaces and A and B two open subsets of X such that $X = A \cup B$. Moreover, let $f\colon A \to Y$ and $g\colon B \to Y$ be continuous functions. If $f(x) = g(x)$ for every $x \in A \cap B$, then the function $h\colon X \to Y$ defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A; \\ g(x) & \text{if } x \in B. \end{cases}$$

is continuous.

Proof. Clearly, h is well-defined. Let now U be an open subset of Y. It is easy to see that $h^{-1}(U) = f^{-1}(U) \cup g^{-1}(U)$. Moreover, since f is continuous, $f^{-1}(U)$ is open in the subspace topology on A, i.e. there exists an open subset V of X such that $f^{-1}(U) = V \cap A$. Therefore, $f^{-1}(U)$ is an open subset of X. Similarly, $g^{-1}(U)$ is an open subset of X and the conclusion follows. ■

The reader should notice that the Pasting Lemma holds even if A and B are both closed.

2.5 Product topology

In addition to the subspace topology, we now introduce another natural way of building new topological spaces:

Definition 2.18 — Product topology. Let $(X_1, \mathscr{T}_1), \ldots, (X_n, \mathscr{T}_n)$ be topological spaces. The product topology for the Cartesian product $X_1 \times \cdots \times X_n$ is the topology with base

$$\mathscr{B} = \{U_1 \times \cdots \times U_n : U_j \in \mathscr{T}_i,\ 1 \le j \le n\}.$$

A set in $\mathscr{B}$ is an elementary open set.

It is easy to check that $\mathscr{B}$ satisfies the properties in Proposition 2.6.

■ **Example 2.31** Consider $\mathbb{R}^n$. The product topology (of $\mathbb{R}$ with the Euclidean topology), the Euclidean topology and the topology induced by the max-norm are the same topologies on $\mathbb{R}^n$, i.e. they have the same open sets. In other words, these three topological spaces are homeomorphic.

We have already seen that the Euclidean topology and the topology induced by the max-norm on $\mathbb{R}^n$ are the same (Example 1.15 and Lemma 1.5). It is therefore enough to show that the product topology $\mathscr{T}$ and the topology induced by the max-norm $\mathscr{T}'$ are the same. In order to show the two containments, we apply the criterion in Exercise 2.11 by taking as a base for $\mathscr{T}'$ the family of open balls.

Let us first show that $\mathscr{T}' \supset \mathscr{T}$. Let $(x_1,\dots,x_n) \in \mathbb{R}^n$ and let B be an element of the base for the product topology such that $(x_1,\dots,x_n) \in B$. This means that $B = U_1 \times \cdots \times U_n$, where each U_i is an open subset of $\mathbb{R}$ (with the Euclidean topology). Therefore, $(x_1,\dots,x_n) \in (x_1-\varepsilon_1, x_1+\varepsilon_1) \times \cdots \times (x_n-\varepsilon_n, x_n+\varepsilon_n)$, for some positive ε_i's. But then, if $\varepsilon = \min_i \varepsilon_i$, we have that the open ball for $\mathscr{T}'$ with center $(x_1,\dots,x_n)$ and radius ε is contained in $(x_1-\varepsilon_1, x_1+\varepsilon_1) \times \cdots \times (x_n-\varepsilon_n, x_n+\varepsilon_n) \subset B$.

Conversely, let $(x_1,\dots,x_n) \in \mathbb{R}^n$ and let B' be an open ball for the topology induced by the max-norm containing $(x_1,\dots,x_n)$. Suppose B' has center $(y_1,\dots,y_n)$ and radius $\varepsilon > 0$. Since B' is exactly the set $(y_1-\varepsilon, y_1+\varepsilon) \times \cdots \times (y_n-\varepsilon, y_n+\varepsilon)$, it is also an element of the base for the product topology $\mathscr{T}$ and so $\mathscr{T} \supset \mathscr{T}'$. ■

For $1 \le j \le n$, the map $\pi_j \colon X_1 \times \cdots \times X_n \to X_j$ defined by $\pi_j((x_1,\dots,x_n)) = x_j$ is the **projection onto the j-th coordinate**. The projections are continuous functions. Indeed, if $U_j \in \mathscr{T}_j$ is an open set of X_j, we have that

$$\pi_j^{-1}(U_j) = X_1 \times \cdots \times X_{j-1} \times U_j \times X_{j+1} \times \cdots \times X_n.$$

Exercise 2.17 Let $X = X_1 \times \cdots \times X_n$ be the product of the topological spaces $X_1,\dots,X_n$. Show that the product topology for X is the coarser topology for which the projections are continuous.
Enter ■

The following important property translates the continuity of a function into a product space to the continuity of its component functions.

Proposition 2.9 — Characteristic property of the product. Let Y be a topological space and $X_1 \times \cdots \times X_n$ a product space. A function $f \colon Y \to X_1 \times \cdots \times X_n$ is continuous if and only if, for each $1 \le j \le n$, the function $\pi_j \circ f : Y \to X_j$ is continuous.

Proof. If f is continuous, then $\pi_j \circ f$ is continuous, being composition of continuous functions.

Conversely, suppose that each $\pi_j \circ f$ is continuous and let $U = U_1 \times \cdots \times U_n$ belong to the base for the product topology. It is easy to see that

$$f^{-1}(U) = (\pi_1 \circ f)^{-1}(U_1) \cap \cdots \cap (\pi_n \circ f)^{-1}(U_n).$$

But then $f^{-1}(U)$ is an open set of Y and we conclude by Proposition 2.7. ■

We now introduce a natural way to build continuous real-valued functions from a certain topological space X given existing ones. If $f, g \colon X \to \mathbb{R}$ are continuous functions, then $f + g \colon X \to$

$\mathbb{R}$, $fg\colon X \to \mathbb{R}$ and $\frac{f}{g}\colon X \to \mathbb{R}$ are defined as follows:

$$\begin{aligned}(f+g)(x) &= f(x)+g(x)\\ fg(x) &= f(x)g(x)\\ \frac{f}{g}(x) &= \frac{f(x)}{g(x)} \quad \text{if } g(x)\neq 0 \text{ for every } x\in X.\end{aligned}$$

Corollary 2.1 Let X be a topological space. If $f,g\colon X \to \mathbb{R}$ are continuous functions, then $f+g$, fg and $\frac{f}{g}$ are continuous as well.

Proof. We just consider the case of $f+g$ and leave the others as an exercise.

Observe first that the function $s\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ defined by $s(a,b) = a+b$ is Lipschitz and so continuous. Indeed, for every $(a,b),(a_0,b_0) \in \mathbb{R}^2$, we have

$$|s(a,b)-s(a_0,b_0)| = |(a-a_0)+(b-b_0)| \le |a-a_0|+|b-b_0| \le 2\sqrt{(a-a_0)^2+(b-b_0)^2}.$$

Moreover, $f+g$ is the composition of s and the function $\phi\colon X \to \mathbb{R}\times\mathbb{R}$ defined by $\phi(x) = (f(x),g(x))$. The characteristic property of the product implies that ϕ is continuous and so $f+g = s\circ\phi$ is continuous by Proposition 2.8. ■

Exercise 2.18 Let $X = X_1\times\cdots\times X_n$ be the product of the topological spaces $X_1,\ldots,X_n$. Show that each projection $\pi_j\colon X \to X_j$ is an open function, i.e. the image of every open set is open. Enter ■

■ **Example 2.32** In general, projections need not be closed functions, i.e. images of closed sets need not be closed.

Indeed, consider $\mathbb{R}^2$ with the Euclidean topology and the set $C = \{(x,y)\in\mathbb{R}^2 : xy = 1\}$. The set C is closed, being the preimage of $\{1\}$ under the continuous function $p\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $p(x,y) = xy$. On the other hand, $\pi_1(C) = \mathbb{R}\setminus\{0\}$, which is not closed. ■

Exercise 2.19 Let X and Y be two topological spaces and consider the product space $X\times Y$. Show that, for every $y\in Y$, the space $X\times\{y\}$ (with the subspace topology) is homeomorphic to X. Enter ■

2.6 Function spaces

In Example 1.12, we have seen how the fundamental notion of uniform convergence of a sequence of real-valued functions on $[0,1]$ can be viewed as convergence in a certain metric space. We now take one step further and define more general functions spaces, i.e. topological spaces whose points are functions. We restrict ourselves to the case of functions from a topological space to a metric space, which contains many intereseting examples and is sufficiently general for our purposes. It turns out there are many different topologies for such function spaces but as in Example 1.12 we will mainly be concerned with an important topology encoding the notion of uniform convergence.

To begin with, as is the case of real-valued functions, we can define pointwise convergence in the context of topological spaces:

Definition 2.19 — Pointwise convergence. Let X be a topological space and (Y,d) a metric space. A sequence of functions $\{f_n\}$ with $f_n\colon X \to Y$ converges pointwise to the function $f\colon X \to Y$ if, for every $x \in X$, the sequence $\{f_n(x)\}$ converges to $f(x)$ in Y.

Uniform convergence is a stronger alternative to pointwise convergence having nicer theoretical properties:

Definition 2.20 — Uniform convergence. Let X be a topological space and (Y,d) a metric space. A sequence of functions $\{f_n\}$ with $f_n\colon X \to Y$ converges uniformly to the function $f\colon X \to Y$ if, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(f_n(x), f(x)) < \varepsilon$ for every $n > N$ and $x \in X$.

Clearly, uniform convergence implies pointwise convergence but the following standard example shows the converse is not true.

▪ **Example 2.33** Consider $[0,1]$ and $\mathbb{R}$ with the Euclidean metric and $f_n\colon [0,1] \to \mathbb{R}$ defined by $f_n(x) = x^n$. We have that $\{f_n\}$ converges pointwise to

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1; \\ 1 & \text{if } x = 1. \end{cases}$$

On the other hand, the convergence is not uniform. Indeed, taking $\varepsilon = 1/100$ and $x_n = \sqrt[n]{1/2}$, we have that there is no N such that $|f_n(x_n) - 0| < \varepsilon$ for every $n \ge N$. [3] ▪

An important property of uniform convergence is that the uniform limit of a convergent sequence of continuous functions is continuous:

Theorem 2.2 — Uniform Limit Theorem. Let X be a topological space, (Y,d) a metric space and $\{f_n\}$ a sequence of continuous functions $f_n\colon X \to Y$. If $\{f_n\}$ converges uniformly to f, then f is continuous.

Proof. Let x_0 be an arbitrary point of X and U be a neighbourhood of $f(x_0)$. We show that $f^{-1}(U)$ is a neighbourhood of x_0. Indeed, choose $\varepsilon > 0$ such that $B_\varepsilon(f(x_0)) \subset U$. Clearly, $f^{-1}(B_\varepsilon(f(x_0))) \subset f^{-1}(U)$. Since $\{f_n\}$ converges uniformly to f, there exists $N \in \mathbb{N}$ such that $d(f_n(x), f(x)) < \varepsilon/3$ for every $n > N$ and $x \in X$. Therefore, let $n > N$. Since f_n is continuous at x_0, we have that $f_n^{-1}(B_{\varepsilon/3}(f_n(x_0)))$ is a neighbourhood of x_0 and so it is enough to show that

$$f_n^{-1}(B_{\varepsilon/3}(f_n(x_0))) \subset f^{-1}(B_\varepsilon(f(x_0))).$$

But for $x \in f_n^{-1}(B_{\varepsilon/3}(f_n(x_0)))$, we have

$$\begin{aligned} d(f(x), f_n(x)) &< \varepsilon/3 && \text{by the choice of } N; \\ d(f_n(x), f_n(x_0)) &< \varepsilon/3 && \text{since } f_n(x) \in B_{\varepsilon/3}(f_n(x_0)); \\ d(f_n(x_0), f(x_0)) &< \varepsilon/3 && \text{by the choice of } N. \end{aligned}$$

Finally, by the triangle inequality, $d(f(x), f(x_0)) < \varepsilon$ and so $x \in f^{-1}(B_\varepsilon(f(x_0)))$. ∎

Example 2.33 shows that a similar statement does not hold for pointwise convergence. Moreover, Theorem 2.2 gives a quick way to see that the sequence in Example 2.33 cannot converge uniformly: if $\{f_n\}$ converges uniformly to f, then it converges pointwise to f and f must be continuous (as the f_n's are).

The function space we want to construct consists of bounded functions:

[3] Note that in the reasoning we are implicitly using uniqueness of limits in a metric space (why?).

Definition 2.21 — Bounded set, bounded function. Let (X,d) be a metric space. A subset A of X is bounded if there exists $M \in \mathbb{R}$ such that $d(a_1,a_2) \leq M$, for every $a_1,a_2 \in X$.

Let X be a topological space and (Y,d) a metric space. A function $f\colon X \to Y$ is bounded if the image $f(X)$ is a bounded subset of Y.

Note that boundedness is not a topological property, as it depends on the particular metric considered.

We can finally define a metric on the set $B(X,Y)$ of bounded functions from X to (Y,d). Since our aim is to encode the notion of uniform convergence, it is natural to consider the function ρ given by $\rho(f,g) = \sup_{x\in X} d(f(x),g(x))$. It clearly satisfies all the properties of a metric.

Definition 2.22 — Uniform metric. Let X be a topological space and (Y,d) a metric space. The uniform metric on $B(X,Y)$ is the metric ρ defined by $\rho(f,g) = \sup_{x\in X} d(f(x),g(x))$.

We denote by $C^0_b(X,Y)$ the subspace of $B(X,Y)$ consisting of the continuous functions. It will turn out that if X has a certain property called compactness, every continuous function from X to Y is bounded and so, in accordance with the familiar case of $C^0([0,1],\mathbb{R})$, we will simply write $C^0(X,Y)$ if this holds.

The restriction to bounded functions is due to the fact that sup might not exist. We just remark, en passant, that this issue can in fact be easily fixed:

Lemma 2.11 — Standard bounded metric. Let (X,d) be a metric space. The map $\overline{d}\colon X\times X \to \mathbb{R}$ defined by $\overline{d}(x,y) = \min\{d(x,y),1\}$ is a metric, the standard bounded metric. Moreover, $(X,d) \cong (X,\overline{d})$.

Proof. We leave to the reader to check that $\overline{d}$ is a metric. The second assertion follows from the fact that, in any metric space, the family of open balls with radius less than 1 is a base for the metric topology (Lemma 2.5). Since this family is the same in (X,d) and $(X,\overline{d})$, the two spaces are homeomorphic. ■

R In the following, $B(X,Y)$ and $C^0_b(X,Y)$ will always be equipped with the uniform metric.

As we would expect, the convergence in $B(X,Y)$ is just the uniform convergence:

Proposition 2.10 Let X be a topological space and (Y,d) a metric space. A sequence of functions $\{f_n\}$ in $B(X,Y)$ converges to f if and only if it converges uniformly to f.

Proof. The proof is exactly the same as the one in Example 1.12 once we show that if $\{f_n\}$ converges uniformly to f, then f is bounded. In order to verify this, observe that uniform convergence implies there exists $N \in \mathbb{N}$ such that $d(f_n(x),f(x)) < 1$ for every $n \geq N$ and $x \in X$. Fix now $n \geq N$. Since f_n is bounded, there exists $M \in \mathbb{R}$ such that $d(f_n(x),f_n(y)) \leq M$, for every $x,y \in X$. Therefore, for every $x,y \in X$, we have

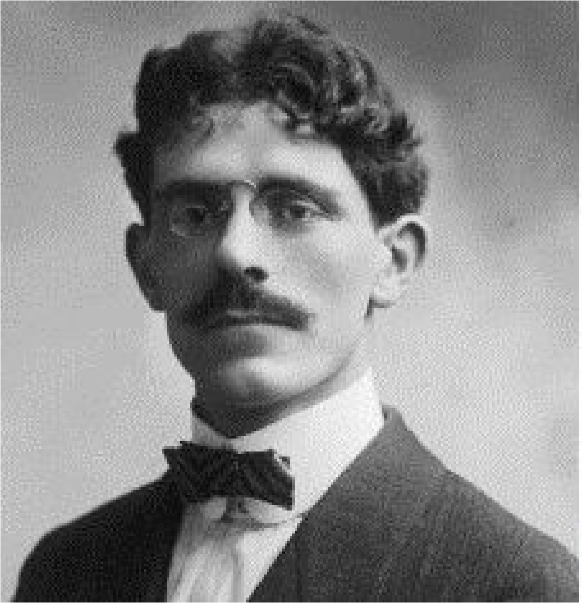
$$\begin{aligned} d(f(x),f(y)) &\leq d(f(x),f_n(x)) + d(f_n(x),f(y)) \\ &\leq d(f(x),f_n(x)) + d(f_n(x),f_n(y)) + d(f_n(y),f(y)) \\ &< M+2, \end{aligned}$$

and so f is bounded. ■

There is a fancy and useful way to rephrase the previous observations:

Corollary 2.2 Let X be a topological space and Y a metric space. $C_b^0(X,Y)$ is a closed subset of $B(X,Y)$.

Proof. By Lemma 1.4, $C_b^0(X,Y)$ is closed if and only if every convergent sequence in $C_b^0(X,Y)$ converges to an element of $C_b^0(X,Y)$. But the convergence in $B(X,Y)$ is just the uniform convergence and so Theorem 2.2 implies that the limit of a sequence in $C_b^0(X,Y)$ is continuous, as desired. ■



3. Complete Metric Spaces

The notion of convergence of a sequence in a metric space clearly depends not only on the sequence itself but also on the limit. The following notion gives us a way to talk about convergence avoiding any reference to the limit:

Definition 3.1 — Cauchy sequence. Let (X,d) be a metric space. A sequence $\{x_n\}$ of points of X is a Cauchy sequence in (X,d) if, for each $\varepsilon > 0$, there exists N such that $d(x_n,x_m) < \varepsilon$ for every $n,m \geq N$.

The metric space (X,d) is complete if every Cauchy sequence in X converges.

Note that every convergent sequence is Cauchy. Indeed, for a given $\varepsilon > 0$, if $\{x_n\}$ converges to x, choose N such that $d(x,x_n) < \varepsilon/2$ for every $n \geq N$. This gives $d(x_n,x_m) \leq d(x_n,x) + d(x,x_m) < \varepsilon$ for every $n,m \geq N$.

In other words, complete metric spaces are those metric spaces for which convergent and Cauchy sequences are the same. In this case, the nice feature is that we know the existence of the limit without any need to explicitly present it.

■ **Example 3.1** The reader is certainly familiar with the fact that $\mathbb{R}$ with the Euclidean metric is complete. On the other hand, the subspace $(0,1)$ is not: just consider the Cauchy sequence $\{1/n\}$. Similarly, the subspace $\mathbb{Q}$ is not complete: just consider any sequence which converges to an irrational number.

Every discrete metric space is complete, as in such a space every Cauchy sequence is eventually constant and so convergent. ■

Lemma 3.1 A metric space (X,d) is complete if every Cauchy sequence in X has a convergent subsequence.

Proof. Let $\{x_n\}$ be a Cauchy sequence in (X,d) and suppose that it admits a subsequence $\{x_{n_i}\}$ convergent to x. We show that $\{x_n\}$ converges to x as well.

Let $\varepsilon > 0$. Since $\{x_n\}$ is Cauchy, we can find N such that $d(x_n,x_m) < \varepsilon/2$ for every $n,m \geq N$. Moreover, since $\{x_{n_i}\}$ converges to x, there exists an index i such that $n_i \geq N$ and $d(x_{n_i},x) < \varepsilon/2$.

Therefore, for $n \geq N$, we have that $d(x_n, x) \leq d(x_n, x_{n_i}) + d(x_{n_i}, x) < \varepsilon$. ∎

It is clear from the definition of equivalence between two metrics d, d' on X (Definition 1.9) that $\{x_n\}$ is a Cauchy sequence in (X,d) if and only if it is a Cauchy sequence in (X,d'). We use this fact to show that the Euclidean metric space is complete:

Proposition 3.1 $\mathbb{R}^k$ with the Euclidean metric (or any other metric equivalent to it) is complete.

Proof. We have seen in Example 1.15 that the Euclidean metric and the metric induced by the max-norm $\|\cdot\|_\infty$ are equivalent and so they give rise to the same Cauchy sequences and to the same convergent sequences (Lemma 1.5). It is therefore enough to show that $\mathbb{R}^k$ with the metric induced by the max-norm is complete.

Let $\{\boldsymbol{x}_n\}$ be a Cauchy sequence in $\mathbb{R}^k$ with $\boldsymbol{x}_n = (x_n^1, \dots, x_n^k)$. Given $\varepsilon > 0$, there exists N such that

$$\|\boldsymbol{x}_n - \boldsymbol{x}_m\|_\infty = \max\{|x_n^1 - x_m^1|, \dots, |x_n^k - x_m^k|\} < \varepsilon,$$

for every $n, m \geq N$. This implies that, for every $1 \leq j \leq k$, $\{x_n^j\}$ is a Cauchy sequence in $\mathbb{R}$ and so, since $\mathbb{R}$ is complete, it converges to some x^j. Therefore, it is easy to see that $\{\boldsymbol{x}_n\}$ converges to $\boldsymbol{x} = (x^1, \dots, x^k)$. ∎

R Completeness is not a topological property. Indeed, we have seen that $\mathbb{R} \cong (0,1)$ but while $\mathbb{R}$ is complete, $(0,1)$ is not. Even more: in contrast to convergent sequences, Cauchy sequences are not preserved under homeomorphisms. Indeed, $\tan\colon (-\pi/2, \pi/2) \to \mathbb{R}$ is a homeomorphism (Example 2.30). On the other hand, even though the sequence $\{y_n\}$ with $y_n = \arctan n$ is Cauchy in $(-\pi/2, \pi/2)$ (Exercise 3.1), the sequence $\{x_n\}$ with $x_n = n$ is clearly not Cauchy in $\mathbb{R}$.

Exercise 3.1 Show that the sequence $\{y_n\}$ with $y_n = \arctan n$ is Cauchy in $(-\pi/2, \pi/2)$. ▪

Exercise 3.2 Show that if $\mathscr{F}$ is a family of complete subspaces of a metric space X, then $\bigcap_{F \in \mathscr{F}} F$ is complete. ▪

We have seen in Example 3.1 that a subspace of a complete metric space need not be complete. However, closedness implies completeness:

Lemma 3.2 A closed subspace Y of a complete metric space X is complete.

Proof. Let $\{y_n\}$ be a Cauchy sequence in Y. Clearly, $\{y_n\}$ is a Cauchy sequence in X as well and so, since X is complete, there exists $x \in X$ such that $\{y_n\}$ converges to x in X. On the other hand, since Y is closed, Lemma 1.4 implies that $x \in Y$. Therefore, $\{y_n\}$ converges to x in Y and Y is complete. ∎

Lemma 3.3 A complete subspace Y of a metric space X is closed in X.

Proof. Let $x \in X$ be an adherent point of Y, i.e. $x \in \overline{Y}$. We show that $x \in Y$. By Proposition 2.4, there exists a sequence $\{y_n\}$ in Y convergent to x. Clearly, $\{y_n\}$ is a Cauchy sequence in Y and so, since Y is complete, it converges to some point $y \in Y$. By uniqueness of limits, we have $x = y$ and so $x \in Y$. ∎

Theorem 3.1 If (Y,d) is a complete metric space, then $B(X,Y)$ and $C_b^0(X,Y)$ are complete.

Proof. Clearly, since $C_b^0(X,Y)$ is a closed subset of $B(X,Y)$, it is enough to show that $B(X,Y)$ is complete.

Therefore, let $\{f_n\}$ be a Cauchy sequence in $B(X,Y)$ and let $\varepsilon > 0$. Since $\{f_n\}$ is Cauchy, there exists $N \in \mathbb{N}$ such that $\rho(f_n, f_m) = \sup_{x\in X} d(f_n(x), f_m(x)) < \varepsilon/3$ for every $n,m \geq N$. This implies that $\{f_n(x)\}$ is a Cauchy sequence in (Y,d) for every fixed $x \in X$, and so, by the completeness of (Y,d), $\{f_n(x)\}$ converges in Y. We can therefore define a function $f\colon X \to Y$ by letting $f(x) = \lim_{n\to\infty} f_n(x)$. We claim that $\{f_n\}$ converges to f in $B(X,Y)$.

We begin by showing that $f_n \to f$. Observe first that, for each $x \in X$, there exists $n_x \geq N$ such that $d(f_{n_x}(x), f(x)) < \varepsilon/3$ (as $f_n(x) \to f(x)$). Therefore, for every $n \geq N$ and $x \in X$, we have

$$d(f(x), f_n(x)) \leq d(f(x), f_{n_x}(x)) + d(f_{n_x}(x), f_n(x)) < 2\varepsilon/3.$$

But this implies that, for every $n \geq N$, we have $\rho(f_n, f) = \sup_{x\in X} d(f_n(x), f(x)) \leq 2\varepsilon/3 < \varepsilon$ and so $f_n \to f$.

It remains to show that f is indeed bounded. Since each f_n is bounded, there exists $M \in \mathbb{R}$ such that $d(f_n(x), f_n(y)) \leq M$, for every $x, y \in X$. Therefore, for every $x, y \in X$, fixing $n \geq N$, we have

$$\begin{aligned}
d(f(x), f(y)) &\leq d(f(x), f_n(x)) + d(f_n(x), f(y)) \\
&\leq d(f(x), f_n(x)) + d(f_n(x), f_n(y)) + d(f_n(y), f(y)) \\
&< M + 4\varepsilon/3,
\end{aligned}$$

and so f is bounded. ■

As the uniform metric and the integral metric on $C^0([0,1],\mathbb{R})$ are not equivalent (Example 1.16), it is interesting to check whether $C^0([0,1],\mathbb{R})$ equipped with the integral metric is complete. It turns out this is not the case:

■ **Example 3.2** The space $C^0([0,1],\mathbb{R})$ with the integral metric (see Example 1.7) is not complete. Indeed, consider the sequence $\{f_n\}_{n\geq 2}$ in $C^0([0,1],\mathbb{R})$ defined as follows:

$$f_n(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 1/2 - 1/n; \\ n(x - 1/2) + 1 & \text{if } 1/2 - 1/n < x \leq 1/2; \\ 1 & \text{if } 1/2 < x \leq 1. \end{cases}$$

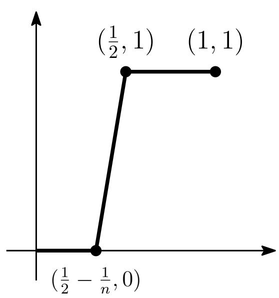


Figure 3.1: The function $f_n\colon [0,1] \to [0,1]$.

Let us first show it is Cauchy. Denoting by d the integral metric, we have

$$\begin{aligned}
d(f_n,f_m) &= \int_0^1 |f_n(t)-f_m(t)|\ \mathrm{d}t \\
&= \int_0^{1/2-1/n} |f_n(t)-f_m(t)|\ \mathrm{d}t + \int_{1/2-1/n}^{1/2} |f_n(t)-f_m(t)|\ \mathrm{d}t + \int_{1/2}^{1} |f_n(t)-f_m(t)|\ \mathrm{d}t \\
&\le \int_{1/2-1/n}^{1/2} |f_n(t)|\ \mathrm{d}t + \int_{1/2-1/n}^{1/2} |f_m(t)|\ \mathrm{d}t \\
&= \frac{1}{2}\left(\frac{1}{n}+\frac{1}{m}\right).
\end{aligned}$$

Suppose now there exists $f \in C^0([0,1],\mathbb{R})$ such that $f_n \to f$, i.e. $d(f_n,f) \to 0$. Observe first that

$$\int_{1/2-1/n}^{1/2} |f(t)|\ \mathrm{d}t \le \int_{1/2-1/n}^{1/2} |f_n(t)-f(t)|\ \mathrm{d}t + \int_{1/2-1/n}^{1/2} |f_n(t)|\ \mathrm{d}t = \int_{1/2-1/n}^{1/2} |f_n(t)-f(t)|\ \mathrm{d}t + \frac{1}{2n}.$$

Therefore,

$$\begin{aligned}
\int_0^{\frac{1}{2}} |f(t)|\ \mathrm{d}t + \int_{\frac{1}{2}}^{1} |1-f(t)|\ \mathrm{d}t &= \int_0^{\frac{1}{2}-\frac{1}{n}} |f(t)|\ \mathrm{d}t + \int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}} |f(t)|\ \mathrm{d}t + \int_{\frac{1}{2}}^{1} |1-f(t)|\ \mathrm{d}t \\
&\le \int_0^{\frac{1}{2}-\frac{1}{n}} |f(t)|\ \mathrm{d}t + \int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}} |f_n(t)-f(t)|\ \mathrm{d}t + \frac{1}{2n} + \int_{\frac{1}{2}}^{1} |1-f(t)|\ \mathrm{d}t \\
&= d(f_n,f) + \frac{1}{2n}.
\end{aligned}$$

Since the RHS tends to 0 as $n \to \infty$ and the LHS is a sum of non-negative terms which is independent of n, it must be $\int_0^{1/2} |f(t)|\ \mathrm{d}t = 0$ and $\int_{1/2}^{1} |1-f(t)|\ \mathrm{d}t = 0$. Therefore, since the function $|f|$ is continuous in $[0,1/2]$ and the function $|1-f|$ is continuous in $[1/2,1]$, we conclude that they both are the identically zero functions in the respective intervals (by the reasoning adopted in Example 1.7). In other words, $f(x)=0$ for each $x \in [0,1/2]$ and $f(x)=1$ for each $x \in (1/2,1]$, a contradiction to the continuity of f. ∎

We mention that there is a general construction for "completing" a metric space, similar to the one in which $\mathbb{R}$ can be obtained from $\mathbb{Q}$. Indeed, given an arbitrary metric space (X,d), there exists a complete metric space $(\hat{X},\hat{d})$ containing X as a dense subset. The idea is to define an equivalence relation on the set of Cauchy sequences in X: two Cauchy sequences $\{x_n\}$ and $\{y_n\}$ are equivalent if $d(x_n,y_n) \to 0$ as $n \to \infty$. The points of $\hat{X}$ are then the equivalence classes of Cauchy sequences (see, e.g. [8]).

We conclude this chapter by introducing the ubiquitous Banach's Fixed Point Theorem, guaranteeing the existence of fixed points for some special functions on complete spaces.

Definition 3.2 — Fixed point. Let X be a non-empty set and $f\colon X \to X$ a function. A point $x \in X$ is a fixed point for f if $f(x)=x$.

■ **Example 3.3** Every continuous function $f\colon [a,b] \to [a,b]$ has at least one fixed point.

Indeed, $f(x)-x$ is a continuous function and since $f(a) \ge a$ and $f(b) \le b$, we have $f(b)-b \le 0 \le f(a)-a$. Therefore, by the Intermediate Value Theorem, there exists $x \in [a,b]$ such that $f(x)-x=0$ and so x is a fixed point for f.

This is the one-dimensional case of the Brouwer's Fixed Point Theorem asserting that every continuous function from a closed ball in $\mathbb{R}^n$ to itself has at least one fixed point. This theorem can be proved combinatorially via the Sperner's Lemma, a result on triangulations of simplices (see, e.g., [5, 10]). ∎

Banach's Fixed Point Theorem has plenty of deep applications (see, e.g., [2, 12]) but we will content ourselves with the following elementary motivation. Solving an equation $f(x) = 0$, for some $f: \mathbb{R} \to \mathbb{R}$, can be viewed as a fixed point problem. Indeed, setting $g(x) = f(x) + x$, we have that x is a zero of f if and only if it is a fixed point for g.

Definition 3.3 — Contraction. Let (X,d) be a metric space. A contraction on X is a Lipschitz function $f: X \to X$ with Lipschitz constant $L < 1$, i.e. a function f such that, $d(f(x), f(y)) \leq L \cdot d(x,y)$, for every $x, y \in X$.

It is easy to see that a contraction f on X can have at most one fixed point. Indeed, if $f(x) = x$ and $f(y) = y$, we have that $d(x,y) = d(f(x), f(y)) \leq L \cdot d(x,y)$, for some $L < 1$. This implies that $d(x,y) = 0$ and so $x = y$.

Theorem 3.2 — Banach's Fixed Point Theorem. A contraction on a complete metric space has exactly one fixed point.

Proof. Let (X,d) be a complete metric space and let f be a contraction on X such that, for every $x, y \in X$,

$$d(f(x), f(y)) \leq L \cdot d(x,y).$$

By the previous remark, it is enough to show the existence of such a fixed point. Therefore, fix an arbitrary point x_1 of X and, for each $n \geq 1$, define $x_{n+1} = f(x_n)$. We claim that $\{x_n\}$ is a Cauchy sequence. Observe first that

$$d(x_2, x_3) = d(f(x_1), f(f(x_1))) \leq d(x_1, f(x_1)) \cdot L = d(x_1, x_2) \cdot L$$

and, by an easy induction, $d(x_n, x_{n+1}) \leq d(x_1, x_2) \cdot L^{n-1}$. Therefore, for $m \geq n+1$, the triangle inequality and the fact that $L < 1$ imply that

$$\begin{aligned} d(x_n, x_m) \leq d(x_n, x_{n+1}) + \cdots + d(x_{m-1}, x_m) &\leq d(x_1, x_2)(L^{n-1} + \cdots + L^{m-2}) \\ &\leq d(x_1, x_2) \cdot L^{n-1} \sum_{i \geq 0} L^i \\ &= d(x_1, x_2) \cdot L^{n-1} \frac{1}{1-L}. \end{aligned}$$

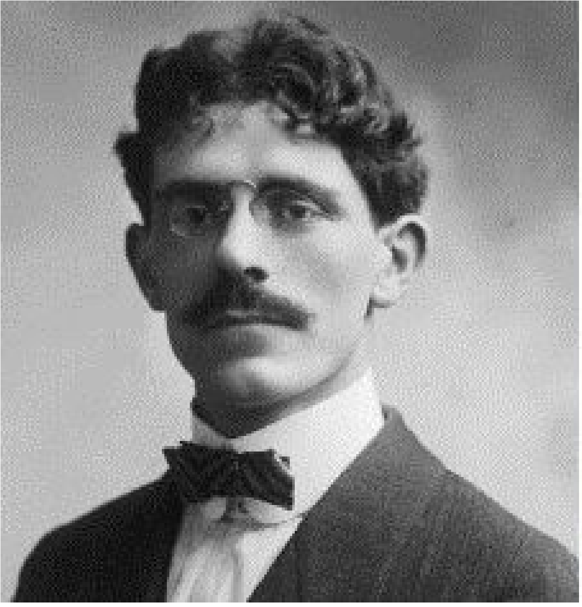
But then $\{x_n\}$ is indeed Cauchy and so, X being complete, it converges to some $x \in X$. Since f is Lipschitz, it is continuous (Example 1.18) and Theorem 1.1 implies that $f(x_n) \to f(x)$. On the other hand, since $x_{n+1} = f(x_n)$, we have that $f(x_n) \to x$ and by uniqueness of limits, we conclude that $f(x) = x$. ■

■ **Example 3.4** Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function such that there exists $k \in (0,1)$ with $|f'(x)| \leq k$ for every $x \in \mathbb{R}$. We claim that f has a unique fixed point. In view of Theorem 3.2 it is enough to show that f is a contraction. But for every $x, y \in \mathbb{R}$ with $x < y$, the Mean Value Theorem implies there exists $c \in (x,y)$ such that $f(y) - f(x) = (y-x)f'(c)$. Therefore, $|f(y) - f(x)| \leq k|y-x|$, as desired. ■

Note that Theorem 3.2 does not hold if the space is not complete. Indeed, consider the subspace $(0,1)$ of $\mathbb{R}$ and the contraction $f: (0,1) \to (0,1)$ given by $f(x) = x/2$.

Moreover, Theorem 3.2 does not hold if the function f just satisfies the weaker condition $d(f(x), f(y)) < d(x,y)$, for every distinct $x, y \in X$. Indeed, consider $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sqrt{x^2+1}$. By the Mean Value Theorem, it is easy to see that $|f(x) - f(y)| < |x-y|$ for every pair of distinct $x, y \in \mathbb{R}$. On the other hand, $f(x) > x$ for each $x \in \mathbb{R}$.

Exercise 3.3 Let (X,d) be a complete metric space and $f\colon X \to X$ be a function such that f^k is a contraction for some $k \geq 1$, where f^k denotes the composition of f with itself k times. Show that f has a unique fixed point. Enter ■



4. Compactness

A fundamental result in Real Analysis is the Extreme Value Theorem: a continuous real-valued function on a closed interval attains its maximum and minimum values. This theorem is blatantly false for general metric spaces and we have seen that the notion of boundedness is not a topological one. In fact, it turns out that the key property involved in the proof of the Extreme Value Theorem is the so-called compactness. This is a completely general notion which is somehow an approximation of finiteness: many arguments which hold for finite sets can be extended to compact sets[1]. Another important aspect of compactness is that it allows to pass from local properties to global ones. We will see this phenomenon throughout the chapter.

Definition 4.1 — Open covering. A family $\mathscr{U}$ of subsets of a space X is a covering of X if the union of the elements of $\mathscr{U}$ is equal to X. The covering is an open covering of X if its elements are open subsets of X.

Definition 4.2 — Compact space. A topological space X is compact if every open covering of X has a finite subcovering, i.e. a finite subfamily which is a covering of X.

A subset of a topological space is compact if it is a compact space with the subspace topology.

Clearly, every space with finitely many points is compact.

■ **Example 4.1** $\mathbb{R}$ with the Euclidean topology is clearly not compact, as the open covering $\{(n,n+2)\}_{n\in\mathbb{Z}}$ has no finite subfamily covering $\mathbb{R}$.

On the other hand, the subspace $X=\{0\}\cup\{1/n : n\in\mathbb{N}\}$ of $\mathbb{R}$ is compact. Indeed, every open covering of X contains an element U containing 0. Moreover, U contains all but finitely many of the points $1/n$ and so, from every open covering of X, we can easily obtain a finite subcovering. ■

■ **Example 4.2** Every subspace of $\mathbb{R}$ with the cofinite topology is compact. Indeed, let X be a subspace of $\mathbb{R}$ and $\mathscr{U}$ an open covering of X. We have that every $U\in\mathscr{U}$ covers all but finitely

[1] Think again about the Extreme Value Theorem: every finite subset of real numbers has a largest and smallest element.

many points of X, as $X \setminus U$ is finite. ■

R The following anecdote is taken from [4]:
There is a story about Sir Michael Atiyah (1929–) and Graeme Segal (1941–) giving an oral exam to a student at Cambridge. Evidently the poor student was a nervous wreck, and it got to a point where he could hardly answer any questions at all. At one point, Atiyah (endeavoring to be kind) asked the student to give an example of a compact set. The student said: “The real line”. Trying to play along, Segal said: “In what topology?”.

■ **Example 4.3** The prototype of a compact subspace of $\mathbb{R}$ (with the Euclidean topology) is the closed interval $[0,1]$. This might already be familiar to the reader but we will reprove it in the next section (together with a generalization).

On the other hand, $[0,1]$ is not a compact subspace of $\mathbb{R}$ with the cocountable topology. Indeed, for every $q \in \mathbb{Q}$, the set $U_q = \mathbb{R} \setminus (\mathbb{Q} \setminus \{q\})$ is open in $\mathbb{R}$ with the cocountable topology and so $\mathscr{U} = \{[0,1] \cap U_q\}_{q \in \mathbb{Q}}$ is an open covering of $[0,1]$. Since $U_q \cap \mathbb{Q} = \{q\}$ and $[0,1]$ contains infinitely many rationals, $\mathscr{U}$ has no finite subcovering. ■

The following two results show that the Hausdorff property and compactness are in some sense “dual” notions:

Lemma 4.1 Let X be a compact topological space. Every closed subspace of X is compact.

Proof. Let Y be a closed subspace of the compact space X, i.e. Y is closed as a subset of X. Let $\mathscr{U}$ be an open covering of the subspace Y. This means that $\mathscr{U} = \{Y \cap U_i\}_{i \in I}$, where each U_i is an open set in X. Since Y is closed, we have that $\mathscr{U}' = \mathscr{U} \cup (X \setminus Y)$ is an open covering of X. The compactness of X implies there exists a finite subfamily of $\mathscr{U}'$ covering X. By removing the set $X \setminus Y$ from the subfamily, we obtain a finite open covering of Y. ■

R In the proof of Lemma 4.1 we have used the little trick of adding $X \setminus Y$ to $\mathscr{U}$, invoking compactness and then removing $X \setminus Y$. This reasoning can be useful in many situations (see [1]):
A father left 17 camels to his three sons and, according to the will, the eldest son should be given a half of all camels, the middle son one-third of all camels and the youngest son one-ninth. This is hard to do but a wise man helped the sons. He added his own camel so that the oldest son took $18/2 = 9$ camels, the middle son $18/3 = 6$ camels and the youngest son $18/9 = 2$ camels. The wise man then took his own camel and went away.

Lemma 4.2 Let X be a Hausdorff topological space. Every compact subspace of X is closed.

Proof. Let Y be a compact subspace of X and x_0 be an arbitrary point in $X \setminus Y$. We show there exists an open neighbourhood of x_0 disjoint from Y. This would imply that $X \setminus Y$ is open and so Y is closed.

Since X is Hausdorff, for each $y \in Y$, there exist open subset U_y and V_y of X such that $x_0 \in U_y$, $y \in V_y$ and $U_y \cap V_y = \varnothing$. Since the family $\{V_y\}_{y \in Y}$ is a covering of Y by open sets of X, there exists a finite subfamily $\{V_{y_1}, \dots, V_{y_n}\}$ covering Y. Observe now that $V = V_{y_1} \cup \cdots \cup V_{y_n}$ is an open set containing Y and disjoint from the open subset $U = U_{y_1} \cap \cdots \cap U_{y_n}$. But then U is the desired open neighbourhood of x_0. ■

R In other words, the proof of Lemma 4.2 shows that we can “separate” points and compact subspaces of a Hausdorff space X by open sets. If X is in addition compact, Lemma 4.1 tells us that we can even separate points and closed subspaces by open sets.

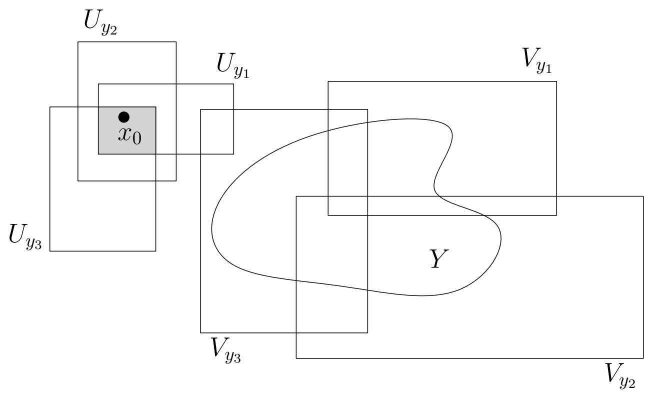


Figure 4.1: Illustration for the proof of Lemma 4.2.

The following result implies that compactness is a topological property:

Lemma 4.3 The image of a compact topological space under a continuous function is compact.

Proof. Let $f\colon X \to Y$ be a continuous function between topological spaces, where X is compact. Let $\mathscr{U}$ be a covering of $f(X)$ by open sets in Y. Since f is continuous, the family $\{f^{-1}(U) : U \in \mathscr{U}\}$ is an open covering of X. Therefore, there exists a finite subfamily $f^{-1}(U_1), \dots, f^{-1}(U_n)$ covering X. Since $f(X) = f(f^{-1}(U_1) \cup \dots \cup f^{-1}(U_n)) \subset U_1 \cup \dots \cup U_n$, we have that $\{U_1, \dots, U_n\}$ is a finite covering of $f(X)$. ■

Corollary 4.1 Let $f\colon X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.

Proof. It suffices to show that images of closed sets of X under f are closed. Therefore, let U be a closed in X. By Lemma 4.1, U is compact and so, by Lemma 4.3, $f(U)$ is compact. Since Y is Hausdorff, Lemma 4.2 implies that $f(U)$ is closed in Y. ■

We can now provide a general version of the Extreme Value Theorem. Recall that the order topology for an ordered set Y is the topology generated by the open intervals, the open rays and Y itself.

Theorem 4.1 — Extreme Value Theorem. Let X be a compact topological space and Y an ordered set with the order topology. If $f\colon X \to Y$ is a continuous function, then there exist points c and d in X such that $f(c) \le f(x) \le f(d)$ for every $x \in X$.

Proof. Since f is continuous and X is compact, the set $A = f(X)$ is compact. We show that A has a largest element M and a smallest element m. Since $m, M \in A$, it must be $m = f(c)$ and $M = f(d)$ for some points $c, d \in X$.

Suppose A has no largest element. This implies that the family $\{(-\infty, a)\}_{a \in A}$ is an open covering of A and so, since A is compact, there exists a finite subfamily $\{(-\infty, a_1), \dots, (-\infty, a_n)\}$ covering A. If a_i is the largest element in $\{a_1, \dots, a_n\}$, then a_i does not belong to any of these sets, a contradiction.

Similarly, it can be shown that A has a smallest element. ■

As with most properties involving open sets, compactness can be checked by looking at bases:

Lemma 4.4 Let X be a topological space and $\mathscr{B}$ be a base for the topology of X. If every open covering of X with sets in $\mathscr{B}$ has a finite subfamily which is a covering, then X is compact.

Proof. Let $\mathscr{U}$ be an open covering of X. For each $x \in X$, there exists $U_x \in \mathscr{U}$ such that $x \in U_x$. On the other hand, by Lemma 2.5, we can find $B_x \in \mathscr{B}$ such that $x \in B_x \subset U_x$. Since $\mathscr{B}' = \{B_x\}_{x\in X}$ is an open covering of X with sets in $\mathscr{B}$, there exists a finite subfamily of $\mathscr{B}'$ which is a covering of X and so a finite subfamily of $\mathscr{U}$ with the same property. ∎

The following important theorem characterizes compactness of product spaces:

Theorem 4.2 — Tychonoff's Theorem, finite case. Let $X_1,\dots,X_n$ be topological spaces. The product space $X_1 \times \cdots \times X_n$ is compact if and only if X_j is compact for each $1 \le j \le n$.

Proof. Suppose first that $X_1 \times \cdots \times X_n$ is compact. Since the projections are continuous functions, Lemma 4.3 implies that each X_j is compact.

Let us show the converse. It is clearly enough to consider the case $n = 2$. Therefore, let X_1 and X_2 be compact spaces. In view of Lemma 4.4, it is enough to consider open coverings with elementary open sets, i.e. sets of the form $U \times V$ with U open of X_1 and V open of X_2. Let $\mathscr{U}$ be such an open covering.

For a fixed $x_2 \in X_2$, we have that $X_1 \times \{x_2\} \cong X_1$ is compact and so there exists a finite subfamily $\{U_1 \times V_1,\dots,U_m \times V_m\}$ of $\mathscr{U}$ covering $X_1 \times \{x_2\}$. By eventually discarding some sets, we may assume that $x_2 \in V_j$, for each $1 \le j \le m$. Therefore $V_{x_2} = V_1 \cap \cdots \cap V_m$ is an open neighbourhood of x_2 in X_2. Moreover, $\pi_2^{-1}(V_{x_2})$ is covered by $\{U_1 \times V_1,\dots,U_m \times V_m\}$.

Since X_2 is compact and the family $\{V_{x_2}\}_{x_2\in X_2}$ is an open covering of X_2, there exists a finite subfamily $\{V_{x_{2,1}},\dots,V_{x_{2,k}}\}$ covering X_2 and so

$$X_1 \times X_2 = \pi_2^{-1}(X_2) = \pi_2^{-1}(V_{x_{2,1}}) \cup \cdots \cup \pi_2^{-1}(V_{x_{2,k}}).$$

Since each $\pi_2^{-1}(V_{x_{2,i}})$ is covered by $\{U_1 \times V_1,\dots,U_m \times V_m\}$, we have that the finite subfamily $\{U_1 \times V_1,\dots,U_m \times V_m\}$ of $\mathscr{U}$ is a covering of $X_1 \times X_2$. ∎

We have the following partial converse of the Uniform Limit Theorem (Theorem 2.2):

Theorem 4.3 — Dini's Theorem. Let X be a compact topological space, let $\{f_n\}$ be a sequence of continuous functions $f_n \colon X \to \mathbb{R}$ such that $f_n(x) \le f_{n+1}(x)$ for every $n \in \mathbb{N}$ and $x \in X$ (i.e., the sequence is monotone increasing) and let $f \colon X \to \mathbb{R}$ be a continuous function.

If $f_n(x) \to f(x)$ for each $x \in X$, then $\{f_n\}$ converges uniformly to f.

Proof. Let $\varepsilon > 0$ and, for each $n \ge 1$, let $U_n = \{x \in X : |f(x) - f_n(x)| < \varepsilon\}$. Note that U_n is open, as both f and f_n are continuous. Moreover, since the sequence $\{f_n\}$ is monotone increasing, we have that $U_n \subset U_{n+1}$, for each $n \ge 1$.

Consider now an arbitrary point $x \in X$. Since $f_n(x) \to f(x)$, there exists N such that $|f(x) - f_n(x)| < \varepsilon$, for every $n \ge N$. Therefore, $X = \bigcup U_n$ and so $\mathscr{U} = \{U_n\}$ is an open covering of X. By compactness, there exists a finite subfamily of $\mathscr{U}$ covering X and so, since $U_n \subset U_{n+1}$ for each $n \ge 1$, there exists N such that $X = U_N$. Therefore, for each $n \ge N$, $U_n = U_N = X$. This implies that $|f(x) - f_n(x)| < \varepsilon$ for every $n \ge N$ and $x \in X$, i.e. $\{f_n\}$ converges uniformly to f. ∎

We now formulate the notion of compactness in terms of closed sets. This is done by introducing the following set-theoretic property:

Definition 4.3 — Finite intersection property. A family $\mathscr{C}$ of subsets of X has the finite intersection property if for every finite subfamily $\{C_1,\dots,C_n\}$ of $\mathscr{C}$, the intersection $C_1\cap\cdots\cap C_n$ is non-empty.

Theorem 4.4 A topological space X is compact if and only if for every family $\mathscr{C}$ of closed subsets of X with the finite intersection property, the intersection $\bigcap_{C\in\mathscr{C}} C$ is non-empty.

Proof. Suppose first X is compact and let $\mathscr{C}$ be a family of closed subsets of X with the finite intersection property. Consider the family of open sets $\mathscr{U}=\{X\setminus C : C\in\mathscr{C}\}$. We claim that no finite subfamily of $\mathscr{U}$ is a covering of X. Indeed, for every finite subfamily $\{X\setminus C_1,\dots,X\setminus C_n\}\subset\mathscr{U}$, we have $X\setminus C_1\cup\cdots\cup X\setminus C_n = X\setminus(C_1\cap\cdots\cap C_n)\neq X$. Since X is compact, $\mathscr{U}$ is not an open covering of X and so $X\setminus\bigcap_{C\in\mathscr{C}} C=\bigcup_{C\in\mathscr{C}} X\setminus C\neq X$. Therefore, $\bigcap_{C\in\mathscr{C}} C\neq\varnothing$.

The other implication follows by reversing the argument above. ■

The formulation of compactness given above allows to pass from the finite version of the Helly's Theorem to the infinite one: If $\mathscr{C}$ is an infinite family of compact convex sets in $\mathbb{R}^d$ such that any $d+1$ of them have non-empty intersection, then all the sets in $\mathscr{C}$ have non-empty intersection (see the beautiful [6]).

The definition of compactness in terms of open coverings allows to obtain simple proofs of powerful theorems. There are two other formulations of compactness having a more analytic flavour and which will turn out to be equivalent in metric spaces (see Theorem 4.9):

Definition 4.4 — Limit point compact. A space X is limit point compact if every infinite subset of X has a limit point.

Definition 4.5 — Sequentially compact. A topological space X is sequentially compact if every sequence of points of X has a convergent subsequence.

Limit point compactness is the weakest among these three forms of compactness:

Proposition 4.1 If a space is compact, then it is limit point compact.

Proof. Let X be a compact space. We show that if a subset has no limit points, then it is finite.

Therefore, suppose $A\subset X$ has no limit points. By Proposition 2.2, A is closed. Moreover, for each $a\in A$, there exists a neighbourhood N_a of a such that $N_a\cap A=\{a\}$ and so an open set U_a such that $a\in U_a$ and $U_a\cap A=\{a\}$. Consider now the open covering $\mathscr{U}$ of X given by $(X\setminus A)\cup\{U_a\}_{a\in A}$. Since X is compact, there exists a finite subfamily of $\mathscr{U}$ covering X and so A is finite. ■

Proposition 4.2 If a space is sequentially compact, then it is limit point compact.

Proof. Let X be a sequentially compact space and A an infinite subset of X. Build a sequence $\{a_n\}$ in A such that $\{a_n : n\in\mathbb{N}\}$ is infinite. By assumption, $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ converging to some $a\in A$. Therefore, a is a limit point of $\{a_n : n\in\mathbb{N}\}$ and so of A. ■

For a general topological space no other relation between compactness, limit point compactness and sequential compactness hold, but it is somehow difficult to come up with the appropriate counterexamples.

4.1 Compactness in Euclidean spaces

In this section we characterize the compact subspaces of $\mathbb{R}^n$ equipped with the Euclidean topology. To begin with, we consider the case $n = 1$ and show that every closed interval is compact. In fact, this can be proved in a more general setting:

Theorem 4.5 Let X be an ordered set having the least upper bound property. Each closed interval in X with the order topology is compact.

Recall that an ordered set X has the least upper bound property if every non-empty subset A of X which is bounded above has a least upper bound, the supremum of A.

Proof. Let a and b be elements of X with $a < b$. Let $\mathscr{U}$ be an open covering of $[a,b]$ and let

$$A = \{x \in [a,b] : [a,x] \text{ can be covered by finitely many elements of } \mathscr{U}\}.$$

We show that $A = [a,b]$. Clearly, $A \neq \varnothing$, as $a \in A$. Moreover, A is bounded above, as $A \subset [a,b]$. Therefore, since X has the least upper bound property, there exists $c = \sup A \in [a,b]$.

Observe now that $c > a$. Indeed, $a \in U$, for some open set $U \in \mathscr{U}$ and so, by Exercise 2.16 and Lemma 2.5, there exists $x > a$ with $a \in [a,x) \subset U$. But then $[a,x]$ can be covered by two elements of $\mathscr{U}$ and so $c \geq x > a$.

We now claim that $A = [a,c]$. Clearly, $A \subset [a,c]$ and so let us show the other containment. Since $c \in [a,b]$ and $c \in U$, for some $U \in \mathscr{U}$, there exists $x < c$ such that $c \in (x,c] \subset U$. On the other hand, there exists $y \in A \cap (x,c]$, for otherwise x is a smaller upper bound for A. But then $[a,c] = [a,y] \cup (x,c]$ can be covered by finitely many elements of $\mathscr{U}$ and so $A \supset [a,c]$.

We finally show that $c = b$. Indeed, suppose that $c < b$. As in the previous paragraph, we have that $c \in (x,y)$, for some $a \leq x < y \leq b$, and there exists $z \in A \cap (x,y)$ (for otherwise, x is a smaller upper bound). But then $[a,y] = [a,z] \cup (z,y]$ can be covered by finitely many elements of $\mathscr{U}$ and so $y > c$ belongs to A, a contradiction. ∎

Since the order topology and the Euclidean topology on $\mathbb{R}$ have the same open sets (Example 2.21), we immediately obtain the following:

Corollary 4.2 Every closed interval in $\mathbb{R}$ is compact.

Theorem 4.6 Let (X,d) be a metric space. If $A \subset X$ is compact, then it is closed and bounded.

Proof. Let A be a compact subspace of X. Since a metric space is Hausdorff, Lemma 4.2 implies that A is closed. Moreover, for a fixed $\varepsilon > 0$, consider the covering $\{B_\varepsilon(x)\}_{x \in X}$ of X by open balls. Since A is compact, there exists a finite subfamily $\{B_\varepsilon(x_1), \dots, B_\varepsilon(x_m)\}$ covering A. Let $M = \max_{\{i,j\} \subset \{1,\dots,m\}} d(x_i,x_j)$ and consider $a_1, a_2 \in A$. We have that $a_1 \in B_\varepsilon(x_i)$ and $a_2 \in B_\varepsilon(x_j)$ for some $1 \leq i,j \leq m$ and so

$$d(a_1,a_2) \leq d(a_1,x_i) + d(x_i,a_2) \leq d(a_1,x_i) + d(x_i,x_j) + d(x_j,a_2) < 2\varepsilon + M.$$

Therefore, A is bounded. ∎

The covering considered in the proof of Theorem 4.6 motivates the following:

Definition 4.6 — Totally bounded space. A metric space (X,d) is totally bounded if, for every $\varepsilon > 0$, there exists a finite covering of X by open balls of radius ε.

As we have seen in the proof above, every totally bounded subspace of a given space is bounded. However, the converse is in general not true. Indeed, $\mathbb{R}$ with the standard bounded metric is clearly bounded but not totally bounded. For otherwise, $\mathbb{R}$ can be covered by finitely many open balls with centers in $S = \{x_1, \ldots, x_m\}$ and radius $1/2$. If $x_j = \max\{x_1, \ldots, x_m\}$ and $x_i \in S$, then $d(x_j + 1, x_i) = \min\{|x_j + 1 - x_i|, 1\} = 1$ and so $x_j + 1 \notin \bigcup_{y \in S} B_{1/2}(y)$.

Exercise 4.1 Show that a totally bounded metric space is separable. Enter ■

Exercise 4.2 Let (X, d) be a metric space. Show that if X is totally bounded, then every $A \subset X$ is totally bounded. Moreover, show that $A \subset X$ is totally bounded if and only if $\overline{A}$ is totally bounded. Enter ■

Theorem 4.7 — Heine-Borel Theorem. A subspace X of $\mathbb{R}^n$ is compact if and only if it is closed and bounded in the Euclidean metric or in the metric coming from the max-norm.

Proof. Note that, by Example 1.15 and Lemma 1.5, the Euclidean metric and the metric coming from the max-norm induce the same topology and have the same bounded sets. Therefore, we just consider the metric coming from the max-norm.

If $X \subset \mathbb{R}^n$ is compact then, by Theorem 4.6, it is closed and bounded. Conversely, suppose that X is closed and bounded in the metric d coming from the max-norm. Boundedness implies there exists M such that, for every $x_1, x_2 \in X$, $d(x_1, x_2) \leq M$. Let x_0 be a fixed point of X and let $a = d(x_0, \mathbf{0})$. For each $x \in X$, we have $d(x, \mathbf{0}) \leq d(x, x_0) + d(x_0, \mathbf{0}) \leq M + a$. Therefore, since the closed ball in the metric d centered at $\mathbf{0}$ and with radius $M + a$ is exactly the set $[-(M+a), M+a]^n$, we have that X is contained in $[-(M+a), M+a]^n$.

On the other hand, Theorem 4.5 implies that $[-(M+a), M+a]$ is compact and so, by Tychonoff's Theorem, $[-(M+a), M+a]^n$ is compact[2]. But then, since X is a closed subset of the compact $[-(M+a), M+a]^n$, it is a compact subspace by Lemma 4.1. ■

4.2 Compactness in metric spaces

In this section, we show that the three formulations of compactness introduced so far are in fact equivalent for metric spaces. A particularly useful tool when dealing with compact metric spaces is the so-called Lebesgue Number Lemma. In order to state it, let us first introduce some terminology.

Definition 4.7 — Diameter. Let (X, d) be a metric space. If A is a bounded and non-empty subset of X, the diameter of A is the quantity $\operatorname{diam}(A) = \sup_{a_1, a_2 \in A} d(a_1, a_2)$.

Lemma 4.5 — Lebesgue Number Lemma. Let $\mathscr{U}$ be an open covering of the metric space (X, d). If X is compact, there exists $\delta > 0$ (called the Lebesgue number of $\mathscr{U}$) such that each subset of X having diameter less than δ is contained in an element of $\mathscr{U}$[a].

[a] Clearly, if $\delta > 0$ is a Lebesgue number of $\mathscr{U}$, then any other $0 < \delta' < \delta$ is.

The idea behind the notion of a Lebesgue number of a covering $\mathscr{U}$ of X is that we know each point $x \in X$ belongs to some $U \in \mathscr{U}$ and this should be true as well for "small" sets containing x,

[2] Note that, by Example 2.31, the product topology for $\mathbb{R}^n$ is the same as the topology coming from the max-norm.

regardless of the "position" of x.

Proof. Let $\mathscr{U}$ be a fixed open covering. If X is an element of $\mathscr{U}$, then every positive number is a Lebesgue number for $\mathscr{U}$. Therefore, suppose $X \notin \mathscr{U}$ and choose a finite subfamily $\{U_1,\dots,U_n\}$ of $\mathscr{U}$ covering X. For each i, let $C_i = X \setminus U_i$ and $f\colon X \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{n}\sum_{i=1}^{n} d(x, C_i).$$

We first show that $f(x) > 0$ for each $x \in X$. Given $x \in X$, choose an index i such that $x \in U_i$ and an $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U_i$. Observe now that, for each $y \in C_i = X \setminus U_i$, we have $d(x,y) \geq \varepsilon$ and so $d(x,C_i) = \inf_{y \in C_i} d(x,y) \geq \varepsilon$. Therefore, $f(x) \geq \varepsilon/n > 0$. Since f is continuous, Theorem 4.1 implies it admits a minimum value δ and we now show that such value is in fact the desired Lebesgue number.

Indeed, consider a subset B of X of diameter less than δ and choose a point $x_0 \in B$. Clearly, $B \subset B_\delta(x_0)$. Moreover, if j is an index such that $\max_{1 \leq i \leq n} d(x_0, C_i) = d(x_0, C_j)$, we have that $\delta \leq f(x_0) \leq d(x_0, C_j)$. Therefore, $B \subset B_\delta(x_0) \subset U_j$. Indeed, if $y \in X \setminus U_j = C_j$, then $d(x_0,y) \geq d(x_0, C_j) \geq \delta$. But then $U_j \in \mathscr{U}$ is the desired open set containing B. ■

It is worth noticing that coverings of non-compact spaces may not have any Lebesgue number: just consider the non-compact subspace $(0,1)$ of $\mathbb{R}$ together with the open covering $\mathscr{U} = \{(1/n, 1)\}_{n \geq 2}$. Suppose δ is a Lebesgue number of $\mathscr{U}$ and consider $n \in \mathbb{N}$ such that $1/n < \delta$. We have that the subset $(0, 1/n)$ of $(0,1)$ has diameter less than δ but is not contained in any element of $\mathscr{U}$.

Using the Lebesgue Number Lemma, we now show that continuous function from a compact metric space satisfy the following stronger property:

Definition 4.8 — Uniformly continuous function. Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f\colon (X, d_X) \to (Y, d_Y)$ is uniformly continuous if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x_1), f(x_2)) < \varepsilon$ for every $x_1, x_2 \in X$ with $d_X(x_1, x_2) < \delta$.

The reader should notice that uniform continuity is a global property, while continuity (Definition 1.10) is a local one.

■ **Example 4.4** A canonical example of uniformly continuous functions is given by Lipschitz functions (Example 1.18). Recall that $f\colon (X, d_X) \to (Y, d_Y)$ is Lipschitz if there exists $L > 0$ such that, for every $x_1, x_2 \in X$, we have $d_Y(f(x_1), f(x_2)) \leq L \cdot d_X(x_1, x_2)$. ■

■ **Example 4.5** The function $f\colon (0, +\infty) \to \mathbb{R}$ given by $f(x) = 1/x$ is continuous but not uniformly continuous. Continuity is clear. Therefore, let $\varepsilon > 0$ and suppose there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for every $x, y \in (0, +\infty)$ with $|x - y| < \delta$. Choose now $x' = \frac{\delta}{1+\delta\varepsilon}$ and $y' = \frac{\delta}{2(1+\delta\varepsilon)}$. Clearly, $|x' - y'| < \delta$ and $|f(x') - f(y')| = \frac{1+\delta\varepsilon}{\delta} > \varepsilon$, a contradiction. ■

Theorem 4.8 — Uniform Continuity Theorem. Let (X, d_X) and (Y, d_Y) be metric spaces with (X, d_X) compact. If $f\colon (X, d_X) \to (Y, d_Y)$ is continuous, then it is uniformly continuous.

Proof. For $\varepsilon > 0$, consider the open covering of Y given by $\{B_{\varepsilon/2}(y)\}_{y \in Y}$. Since f is continuous, $\mathscr{U} = \{f^{-1}(B_{\varepsilon/2}(y))\}_{y \in Y}$ is an open covering of X. Since X is compact, Lemma 4.5 guarantees the existence of a Lebesgue number δ for $\mathscr{U}$. But then, if x_1 and x_2 are two points of X such that $d_X(x_1, x_2) < \delta$, the set $\{x_1, x_2\}$ has diameter less than δ and so it is contained in an element of $\mathscr{U}$, i.e. $\{f(x_1), f(x_2)\} \subset B_{\varepsilon/2}(y)$ for some $y \in Y$. By the triangle inequality, we have $d_Y(f(x_1), f(x_2)) < \varepsilon$ and so f is uniformly continuous. ■

Exercise 4.3 Let (X,d_X) and (Y,d_Y) be metric spaces with (X,d_X) compact. Show that if $f\colon (X,d_X) \to (Y,d_Y)$ is a bijective continuous function, then f^{-1} is uniformly continuous.
Enter

We can finally prove the following characterization of compactness in metric spaces:

Theorem 4.9 Let (X,d) be a metric space. The following are equivalent:
(i). X is compact.
(ii). X is limit point compact.
(iii). X is sequentially compact.

Proof. (i) $\Rightarrow$ (ii) Proposition 4.1.

(ii) $\Rightarrow$ (iii) Consider a sequence $\{x_n\}$ in the compact space X and the set $A = \{x_n : n \in \mathbb{N}\}$ of the values of the sequence. If A is finite, there exists a point x such that $x = x_n$ for infinitely many values of n and so the sequence has a constant subsequence, which clearly converges.

On the other hand, if A is infinite, then it has a limit point x. We construct inductively a subsequence convergent to x as follows. First, choose n_1 such that $x_{n_1} \in B_1(x)$. Then, given n_{i-1}, choose an index $n_i > n_{i-1}$ such that $x_{n_i} \in B_{1/i}(x)$. The existence of such an index follows from the fact that $B_{1/i}(x)$ intersects A in infinitely many points, for otherwise the ball with center x and radius $\min\{d(x,a) : a \in A \cap B_{1/i}(x)\}$ does not intersect $A \setminus \{x\}$[3]. The subsequence $\{x_{n_i}\}$ clearly converges to x.

(iii) $\Rightarrow$ (i) We first show two auxiliary results. We begin with an observation related to Lemma 4.5:

Every open covering of a sequentially compact metric space admits a Lebesgue number.

Indeed, let X be a sequentially compact metric space and $\mathscr{U}$ an open covering of X. Suppose, to the contrary, there is no $\delta > 0$ such that each set of diameter less than δ is contained in an element of $\mathscr{U}$. In particular, for each positive integer n, there exists a non-empty set C_n of diameter less than $1/n$ not contained in any element of $\mathscr{U}$. For each n, choose $x_n \in C_n$ and consider the sequence $\{x_n\}$. By assumption, it contains a subsequence $\{x_{n_i}\}$ convergent to some point $x \in X$. But then $x \in U$, for some open set $U \in \mathscr{U}$, and we can find $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U$. For a large enough index n_i, we have $1/n_i < \varepsilon/2$ and $d(x_{n_i},x) < \varepsilon/2$. Therefore, $C_{n_i} \subset B_{1/n_i}(x_{n_i}) \subset B_{\varepsilon/2}(x_{n_i}) \subset B_\varepsilon(x) \subset A$, a contradiction. $\diamond$

Every sequentially compact metric space is totally bounded.

Let X be a sequentially compact metric and suppose, to the contrary, there exists $\varepsilon > 0$ such that X cannot be covered by finitely many open balls of radius ε. We recursively construct a sequence $\{x_n\}$ in X as follows. First, let $x_1 \in X$ be arbitrary. Since $B_\varepsilon(x_1)$ does not cover X, choose $x_2 \in X \setminus B_\varepsilon(x_1)$. Given $x_1,\dots,x_n$, choose $x_{n+1} \in X \setminus (B_\varepsilon(x_1) \cup \cdots \cup B_\varepsilon(x_n))$. Note that $d(x_{n+1},x_i) \geq \varepsilon$, for each $1 \leq i \leq n$. Therefore, $\{x_n\}$ has no convergent subsequence, since every open ball of radius $\varepsilon/2$ contains at most one x_n, a contradiction. $\diamond$

We can finally conclude our proof: If X is a sequentially compact metric space, then it is compact. Let $\mathscr{U}$ be an open covering of X. We have seen $\mathscr{U}$ admits a Lebesgue number δ. For $\varepsilon = \delta/3$, we can find a finite covering of X by open balls of radius ε. Moreover, each of these balls is contained in an element of $\mathscr{U}$ (as they have diameter $2\delta/3 < \delta$). Therefore, choosing one such element for each of them, we obtain a finite subfamily covering X. ■

[3] Alternatively, simply use Proposition 2.5.

There is yet another characterization of compactness in metric spaces. Theorem 4.9 and Lemma 3.1 imply that every compact metric space is complete. The converse is blatantly false (just consider $\mathbb{R}$ with the Euclidean metric). On the other hand, every compact metric space is totally bounded (see Theorem 4.6) and it turns out that completeness and totally boundedness are sufficient conditions:

Theorem 4.10 A metric space (X,d) is compact if and only if it is complete and totally bounded.

Proof. Suppose first (X,d) is compact. By Theorem 4.9, it is sequentially compact and so Lemma 3.1 implies it is complete. Moreover, for every $\varepsilon > 0$, the open covering $\{B_\varepsilon(x)\}_{x\in X}$ of X admits a finite subcovering and so X is totally bounded.

Conversely, let X be complete and totally bounded. In view of Theorem 4.9, we show that X is sequentially compact. Therefore, let $\{x_n\}$ be a sequence in X. It is enough to construct a subsequence which is Cauchy.

Consider a covering of X with open balls of radius 1, whose existence is guaranteed by totally boundedness. At least one of the balls in this covering contains infinitely many terms of the sequence, say B_1 is such a ball, and let $J_1 = \{n : x_n \in B_1\}$. Consider now a covering of X by finitely many open balls with radius $1/2$. Since J_1 is infinite, at least one of the balls in this new covering contains infinitely many terms with indices in J_1. Say such a ball is B_2. We then let $J_2 = \{n : n \in J_1,\ x_n \in B_2\}$. In general, given an infinite set J_k, we let J_{k+1} to be an infinite subset of J_k such that there exists a ball B_{k+1} of radius $1/(k+1)$ containing x_n for every $n \in J_{k+1}$.

We now build the desired subsequence by choosing indices from the J_i's. Let $n_1 \in J_1$ be arbitrary. Given n_k, choose $n_{k+1} \in J_{k+1}$ such that $n_{k+1} > n_k$ (such an index exists since J_{k+1} is infinite). Since $J_1 \supset J_2 \supset \cdots$, we have that, for $i, j \geq k$, the indices n_i and n_j both belong to J_k. This means that, for all $i, j \geq k$, the points x_{n_i} and x_{n_j} are both contained in a ball B_k of radius $1/k$. Therefore, $\{x_{n_i}\}$ is a Cauchy sequence. ■

Corollary 4.3 Let (X,d) be a complete metric space. A subset of X is compact if and only if it is closed and totally bounded.

Proof. It follows from the fact that a closed subspace of a complete metric space is complete (Lemma 3.2) and that a complete subspace of a metric space is closed (Lemma 3.3). ■

By Exercise 4.2, we immediately obtain the following:

Corollary 4.4 Let (X,d) be a complete metric space. A subset of X has compact closure if and only if it is totally bounded.

4.3 Compactness in function spaces

In this section, we study compact subspaces of the metric space $C^0(X,\mathbb{R}^n)$ (equipped with the uniform metric), where X is a compact topological space. It is worth recalling that, in a metric space, compactness and sequential compactness are equivalent notions. Moreover, we have seen that a subspace of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. It turns out that in the case of $C^0(X,\mathbb{R}^n)$ the situation is more complicated. Indeed, despite the fact that closeness and boundedness are necessary conditions for compactness (Theorem 4.6), they are not sufficient anymore.

Consider for example the metric space $C^0([0,1],\mathbb{R})$ and denote by 0_f the identically zero function in $C^0([0,1],\mathbb{R})$. Let $\overline{B_1}(0_f) = \{g \in C^0([0,1],\mathbb{R}) : \rho(g,0_f) \leq 1\}$ be the closed ball centered at 0_f with radius 1. It is clearly closed and bounded but it is not sequentially compact (or, equivalently, compact). Indeed, let $\{f_n\}$ be the sequence in $\overline{B_1}(0_f)$ with $f_n(x) = x^n$, for each $x \in [0,1]$, and suppose it has a convergent subsequence $\{f_{n_k}\}$. Since $\{f_{n_k}\}$ converges to the same limit as $\{f_n\}$, it must converge to

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1; \\ 1 & \text{if } x = 1. \end{cases}$$

as we have seen in Example 2.33. But f does not belong to $C^0([0,1],\mathbb{R})$, a contradiction.

The problem with the sequence $\{f_n\}$ above is that it oscillates too much. Indeed, an additional necessary condition for a subspace of $C^0(X,\mathbb{R}^n)$ to be compact is that its elements are somehow all close to each other. This is formalized by the notion of equicontinuity:

Definition 4.9 — Equicontinuous family. Let X be a topological space, (Y,d) a metric space and $x_0 \in X$. A family of functions $\mathscr{F}$ in $C^0(X,Y)$ is equicontinuous at x_0 if, for every $\varepsilon > 0$, there exists a neighbourhood U of x_0 such that $d(f(x), f(x_0)) < \varepsilon$ for every $x \in U$ and $f \in \mathscr{F}$.

$\mathscr{F}$ is equicontinuous if it is equicontinuous at each point of X.

Note that continuity of a function $f\colon X \to Y$ at $x_0 \in X$ means that, for every $\varepsilon > 0$, there exists a neighbourhood U of x_0 such that $d(f(x), f(x_0)) < \varepsilon$ for every $x \in U$. Therefore, the point of the definition above is that the same neighbourhood U can be chosen for all the functions in $\mathscr{F}$. In particular, every finite family $\mathscr{F}$ of continuous functions is equicontinuous. Moreover, every subfamily of an equicontinuous family is equicontinuous.

Example 4.6 The family $\{f_n : n \in \mathbb{N}\} \subset C^0([0,1],\mathbb{R})$ with $f_n(x) = x^n$ is not equicontinuous at $x_0 = 1$.

Indeed, suppose it is equicontinuous at $x_0 = 1$. This means that, given $\varepsilon = 1/2$, there exists $\delta > 0$ such that $|x^n - 1| < 1/2$ for every $x \in [0,1] \cap (1-\delta, 1]$ and every $n \in \mathbb{N}$. But taking $x \in (1-\delta, 1)$, we can find $n \in \mathbb{N}$ such that $x^n < 1/2$, a contradiction. ■

Let us immediately show that equicontinuity is indeed a necessary condition for compactness in $C^0(X,\mathbb{R}^n)$. Recall that a metric space is compact if and only if it is complete and totally bounded (Theorem 4.10).

Proposition 4.3 Let X be a compact topological space and let (Y,d) be a metric space. If the family $\mathscr{F}$ in $C^0(X,Y)$ is totally bounded (under the uniform metric ρ), then $\mathscr{F}$ is equicontinuous (under d).

Proof. Suppose $\mathscr{F} \subset C^0(X,Y)$ is totally bounded and let $x_0 \in X$ and $\varepsilon > 0$. We know there exists a finite covering $\{B_{\varepsilon/3}(f_1), \dots, B_{\varepsilon/3}(f_k)\}$ of $\mathscr{F}$ by open balls in the uniform metric ρ. Since each f_i is continuous, we can choose a neighbourhood U of x_0 such that $d(f_i(x), f_i(x_0)) < \varepsilon/3$ for every $x \in U$ (for each fixed i there exists such a neighbourhood U_i and taking the finite interesection of the U_i's we obtain a feasible U).

We now show that for every $x \in U$ and $f \in \mathscr{F}$, we have $d(f(x), f(x_0)) < \varepsilon$, thus concluding the proof. Indeed, let $x \in U$ and $f \in \mathscr{F}$. Clearly, $f \in B_{\varepsilon/3}(f_i)$, for some i. Therefore,

$$\begin{aligned} d(f(x), f_i(x)) &< \varepsilon/3 && \text{since } \rho(f,f_i)<\varepsilon/3; \\ d(f_i(x), f_i(x_0)) &< \varepsilon/3 && \text{since } x \in U; \\ d(f_i(x_0), f(x_0)) &< \varepsilon/3 && \text{since } \rho(f,f_i)<\varepsilon/3; \end{aligned}$$

and so $d(f(x), f(x_0)) < \varepsilon$. ■

Let us now pause for a moment and make some further observations related to equicontinuity when X is a metric space. In this case, it is instructive to compare the notion of a family of uniformly continuous functions with that of an equicontinuous family. Recall that $f\colon (X,d_X) \to (Y,d_Y)$ is uniformly continuous if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x_1), f(x_2)) < \varepsilon$ for every $x_1, x_2 \in X$ with $d_X(x_1,x_2) < \delta$. We have seen in Theorem 4.8 that every continuous function on a compact metric space is uniformly continuous. It turns out that a similar phenomenon occurs for equicontinuity:

Theorem 4.11 — Uniform equicontinuity theorem. Let (X,d_X) and (Y,d_Y) be metric spaces with (X,d_X) compact. If $\mathscr{F} \subset C^0(X,Y)$ is equicontinuous, it satisfies the following property:

For each $\varepsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x_1), f(x_2)) < \varepsilon$ for every $x_1, x_2 \in X$ with $d_X(x_1,x_2) < \delta$ and $f \in \mathscr{F}$.

A family $\mathscr{F} \subset C^0(X,Y)$ satisfying the property above is called **uniformly equicontinuous**. Therefore, uniform equicontinuity of $\mathscr{F} \subset C^0(X,Y)$ might be viewed as "uniform (in $f \in \mathscr{F}$) uniform (in $x \in X$) continuity".

Proof of Theorem 4.11. Let $\varepsilon > 0$. By equicontinuity, for each $x \in X$, there exists an open ball $B_{r_x}(x)$ such that $d_Y(f(x), f(y)) < \varepsilon/2$ for every $y \in B_{r_x}(x)$ and $f \in \mathscr{F}$. The family $\mathscr{U} = \{B_{r_x}(x)\}_{x\in X}$ is an open covering of X and so, since X is compact, let $\delta > 0$ be a Lebesgue number of $\mathscr{U}$ (Lemma 4.5). Therefore, if $x, y \in X$ are such that $d_X(x,y) < \delta$, then they are contained in an open ball $B_{r_z}(z)$, for some $z \in X$. This implies that $d_Y(f(x), f(z)) < \varepsilon/2$ and $d_Y(f(y), f(z)) < \varepsilon/2$ for every $f \in \mathscr{F}$ and so $d_Y(f(x), f(y)) < \varepsilon$ for every $f \in \mathscr{F}$. ■

An example of a uniformly equicontinuous family is given by a set of Lipschitz functions having the same Lipschitz constant. Observe also that every equicontinuous family $\mathscr{F} \subset C^0(X,Y)$ (with (X,d_X) compact) consists of uniformly continuous functions. The converse does not hold:

■ **Example 4.7** A family of uniformly continuous functions is not necessarily equicontinuous. Indeed, let $\{f_n\}$ with $f_n\colon [0,2\pi] \to \mathbb{R}$ defined by $f_n(x) = \sin(nx)$.

We already know by Theorem 4.8 that each f_n is uniformly continuous. We could also reason as follows. Let $\varepsilon > 0$. Since $|f_n'(x)| \le n$ for each $x \in [0,2\pi]$, the Mean-Value Theorem implies that $|f_n(x_1) - f_n(x_2)| \le n|x_1 - x_2|$ for every $x_1, x_2 \in [0,2\pi]$. But then $|f_n(x_1) - f_n(x_2)| < \varepsilon$ for every $x_1, x_2 \in [0,2\pi]$ with $|x_1 - x_2| < \varepsilon/n$.

On the other hand, $\{f_n\}$ is not equicontinuous. Indeed, let $x_0 = 0$ and $0 < \varepsilon < 1$. For every $\delta > 0$, we can choose n_δ such that $x = \pi/2n_\delta < \delta$. This implies that $|x - x_0| < \delta$ but $|f_{n_\delta}(x) - f_{n_\delta}(x_0)| = \sin(\pi/2) = 1 > \varepsilon$. ■

Exercise 4.4 Let $\mathscr{F}$ be a family of differentiable functions $f\colon \mathbb{R} \to \mathbb{R}$ such that, for each $x \in \mathbb{R}$, there exists a neighbourhood U of x and $M > 0$ with $|f'(x)| \le M$ for every $f \in \mathscr{F}$ and $x \in U$.

Show that $\mathscr{F}$ is equicontinuous. Enter ■

Coming back to the problem of characterizing compact subspaces of $C^0(X,\mathbb{R}^n)$, we can finally formulate the first version of Ascoli-Arzelà Theorem. It essentially states that the three necessary conditions we have encountered so far (closedness, boundedness and equicontinuity) are also sufficient:

Theorem 4.12 — Ascoli-Arzelà Theorem, first version. Let X be a compact topological space and let $(\mathbb{R}^n, d)$ be the Euclidean metric space. A subspace $\mathscr{F}$ of $C^0(X,\mathbb{R}^n)$ (with the uniform metric) is compact if and only if it is closed, bounded (under ρ) and equicontinuous (under

d).

Instead of directly proving Theorem 4.12, we will derive it from a characterization of subspaces of $C^0(X,\mathbb{R}^n)$ having compact closure (Ascoli-Arzelà Theorem, second version). We have already seen in Corollary 4.4 that a subset of a complete metric space has compact closure if and only if it is totally bounded. Since checking totally boundedness (and boundedness) is in general not an easy task, in our second version of Ascoli-Arzelà Theorem we look for equivalent conditions which might be easier to handle. It turns out that we can indeed substitute totally boundedness with equicontinuity plus pointwise boundedness:

Definition 4.10 — Pointwise bounded family. Let X be a topological space, (Y,d_Y) a metric space and let $\mathscr{F}$ a family of functions $f\colon X\to Y$. The family $\mathscr{F}$ is pointwise bounded if, for every $x\in X$, the set $\mathscr{F}_x=\{f(x) : f\in\mathscr{F}\}$ is bounded (under d_Y).

It is easy to see that every bounded family is pointwise bounded and we have the following implications:

$$\text{totally bounded} \implies \text{bounded} \implies \text{pointwise bounded}.$$

We can finally state the second version:

Theorem 4.13 — Ascoli-Arzelà Theorem, second version. Let X be a compact topological space and let $(\mathbb{R}^n,d)$ be the Euclidean metric space. A subspace $\mathscr{F}$ of $C^0(X,\mathbb{R}^n)$ (with the uniform metric) has compact closure if and only if $\mathscr{F}$ is equicontinuous and pointwise bounded (under d).

The first version immediately follows from the second:

Proof of Theorem 4.12. If $\mathscr{F}$ is compact, then it is closed and bounded (Theorem 4.6) and equicontinuous as well (Theorem 4.13).

Conversely, if $\mathscr{F}$ is closed, it coincides with its closure. If it is bounded (under ρ), it is pointwise bounded (under d). If it is in addition equicontinuous (under d), Theorem 4.13 implies it is compact. ■

The main feature of the second version with respect to the first is that we have replaced boundedness with the formally weaker condition of pointwise boundedness, often easier to check.

Let us now pass to the proof of Theorem 4.13. Showing necessity is easy:

Proof of Theorem 4.13, $\Rightarrow$. Let $\mathscr{G}$ denote the closure of $\mathscr{F}$ in $C^0(X,\mathbb{R}^n)$. We have to show that if $\mathscr{G}$ is compact, then $\mathscr{F}$ is equicontinuous and pointwise bounded (under d). Since $\mathscr{F}\subset\mathscr{G}$, it is enough to show that $\mathscr{G}$ is equicontinuous and pointwise bounded.

Observe that the compactness of $\mathscr{G}$ implies it is totally bounded (Theorem 4.10) and so, by Proposition 4.3, equicontinuous. Moreover, totally boundedness implies that $\mathscr{G}$ is bounded and so pointwise bounded. Indeed, if $\rho(f,g)\le M$ for every $f,g\in\mathscr{G}$, then $d(f(x),g(x))\le M$ for every $x\in X$ and $f,g\in\mathscr{G}$, and therefore $\mathscr{G}_x=\{f(x) : f\in\mathscr{G}\}$ is bounded. ■

The proof of sufficiency is more involved and requires several steps. Denoting by $\mathscr{G}$ the closure of $\mathscr{F}$ in $C^0(X,\mathbb{R}^n)$, we have to show that if $\mathscr{F}$ is equicontinuous and pointwise bounded (under d), then $\mathscr{G}$ is compact. By Theorem 4.10, it is enough to show that $\mathscr{G}$ is complete and totally bounded (under the uniform metric ρ). The steps are as follows:

1. $\mathscr{G}$ is complete (easy by Lemma 3.2);
2. $\mathscr{G}$ is equicontinuous and pointwise bounded (under d);
3. There exists a compact subspace Y of $\mathbb{R}^n$ such that $\mathscr{G}\subset C^0(X,Y)$;

4. $\mathscr{G}$ is totally bounded.

To obtain 4. from 3. and 2., we prove the following partial converse of Proposition 4.3:

Proposition 4.4 Let X be a compact topological space and let (Y,d) be a compact metric space. If the family $\mathscr{F}$ in $C^0(X,Y)$ is equicontinuous (under d), then $\mathscr{F}$ is totally bounded (under the uniform metric ρ).

Proof. Suppose $\mathscr{F} \subset C^0(X,Y)$ is equicontinuous. We show that, for every $\varepsilon > 0$, there exists a finite covering of $\mathscr{F}$ by open balls of radius ε in the uniform metric ρ.

Therefore, consider $\varepsilon > 0$. By equicontinuity, for any $x \in X$, there exists a neighbourhood U_x of x such that $d(f(x'), f(x)) < \varepsilon/3$ for every $x' \in U_x$ and $f \in \mathscr{F}$. Clearly, we may assume each U_x is open and so the family $\{U_x\}_{x\in X}$ is an open covering of X. By compactness, there exists a finite subfamily $\{U_{x_1}, \dots, U_{x_k}\}$ covering X and to simplify the notation let $U_i = U_{x_i}$. Since Y is compact as well, we can find finitely many open sets $V_1, \dots, V_m$ of diameter less than $\varepsilon/3$ covering Y.

Clearly the set of maps from $\{1,\dots,k\}$ to $\{1,\dots,m\}$ is finite and consider such a map $\alpha\colon \{1,\dots,k\} \to \{1,\dots,m\}$. If there exists $f \in \mathscr{F}$ such that $f(x_i) \in V_{\alpha(i)}$ for each $i \in \{1,\dots,k\}$, we label it f_α and say it represents α (the choice of f is arbitrary). Therefore, $\{f_\alpha\}$ is a finite family. We claim that $\{B_\varepsilon(f_\alpha)\}$ is the desired covering of $\mathscr{F}$ by open balls in the uniform metric ρ.

Indeed, let $f \in \mathscr{F}$ and, for each $i \in \{1,\dots,k\}$, choose an integer $\alpha(i)$ such that $f(x_i) \in V_{\alpha(i)}$. We show that f belongs to the ε-ball with center the representative of α: more precisely, we show that $f \in B_\varepsilon(f_\alpha)$ or, equivalently, $\rho(f, f_\alpha) = \sup_{x\in X}\{d(f(x), f_\alpha(x))\} < \varepsilon$. Therefore, let $x \in X$ and choose an index i such that $x \in U_i$. We have that

$$\begin{aligned} d(f(x), f(x_i)) &< \varepsilon/3 && \text{since } x\in U_i; \\ d(f(x_i), f_\alpha(x_i)) &< \varepsilon/3 && \text{since } f(x_i), f_\alpha(x_i) \in V_{\alpha(i)}; \\ d(f_\alpha(x_i), f_\alpha(x)) &< \varepsilon/3 && \text{since } x\in U_i. \end{aligned}$$

Therefore, $d(f(x), f_\alpha(x)) < \varepsilon$ for every $x \in X$ and so $\rho(f, f_\alpha) < \varepsilon$, as desired. ∎

We can finally conclude the proof of the second version of Ascoli-Arzelà Theorem:

Proof of Theorem 4.13, $\Leftarrow$. Let $\mathscr{G}$ denote the closure of $\mathscr{F}$ in $C^0(X,\mathbb{R}^n)$. We have to show that if $\mathscr{F}$ is equicontinuous and pointwise bounded (under d), then $\mathscr{G}$ is compact. In view of Theorem 4.10, it is enough to show that $\mathscr{G}$ is complete and totally bounded (under the uniform metric ρ). Since $\mathscr{G}$ is a closed subspace of the complete metric space $C^0(X,\mathbb{R}^n)$ (Exercise 2.9), it is complete (Lemma 3.2). Let us finally show total boundedness. We proceed with the following two observations:

$\mathscr{G}$ is equicontinuous and pointwise bounded (under d).

Let us prove equicontinuity first. Let $x_0 \in X$ and $\varepsilon > 0$. Since $\mathscr{F}$ is equicontinuous, there exists a neighbourhood U of x_0 such that $d(f(x), f(x_0)) < \varepsilon/3$ for every $x \in U$ and $f \in \mathscr{F}$. Consider now an arbitrary $g \in \mathscr{G}$. Since g belongs to the closure of $\mathscr{F}$, every neighbourhood of g intersects $\mathscr{F}$ and so there exists $f \in \mathscr{F}$ such that $\rho(f,g) < \varepsilon/3$. But then, for every $x \in U$, we have

$$d(g(x), g(x_0)) \le d(g(x), f(x)) + d(f(x), g(x_0)) \le d(g(x), f(x)) + d(f(x), f(x_0)) + d(f(x_0), g(x_0)) < \varepsilon.$$

This implies that $\mathscr{G}$ is equicontinuous.

As for pointwise boundedness, consider $x \in X$. Since $\mathscr{F}$ is pointwise bounded, there exists M such that the diameter of $\mathscr{F}_x = \{f(x) : f \in \mathscr{F}\}$ is at most M. For $g, g' \in \mathscr{G}$, we now choose $f, f' \in \mathscr{F}$ such that $\rho(f,g) < 1$ and $\rho(f',g') < 1$. This gives

$$d(g(x),g'(x)) \le d(g(x),f(x)) + d(f(x),g'(x)) \le d(g(x),f(x)) + d(f(x),f'(x)) + d(f'(x),g'(x)) < M+2.$$

Therefore, $\mathscr{G}_x = \{g(x) : g \in \mathscr{G}\}$ is bounded for every $x \in X$ and so $\mathscr{G}$ is pointwise bounded. $\diamond$

There exists a compact subspace Y of $\mathbb{R}^n$ containing $\bigcup_{g \in \mathscr{G}} g(X)$.

Let $x \in X$. By the previous observation, $\mathscr{G}$ is equicontinuous and so there exists a neighbourhood U_x of x such that $d(g(x'),g(x)) < 1$ for every $x' \in U_x$ and $g \in \mathscr{G}$. By eventually taking subsets, we may assume U_x is open for every $x \in X$. Therefore, since X is compact, it admits a finite open covering $\{U_{x_1}, \ldots, U_{x_k}\}$. Since $\mathscr{G}$ is pointwise bounded, every $\mathscr{G}_{x_i} = \{g(x_i) : g \in \mathscr{G}\}$ is bounded and so $\bigcup_{i=1}^k \mathscr{G}_{x_i}$ is bounded as well, say it is contained in the ball centered at the origin with radius N (in $\mathbb{R}^n$). Moreover, if $x \in X$, then $x \in U_{x_j}$ for some j and $d(g(x),g(x_j)) < 1$ for each $g \in \mathscr{G}$. Therefore, $d(g(x),\mathbf{0}) \le d(g(x),g(x_j)) + d(g(x_j),\mathbf{0}) < 1+N$ and $\bigcup_{g \in \mathscr{G}} g(X)$ is contained in the open ball $B_{N+1}(\mathbf{0})$. The closure of $B_{N+1}(\mathbf{0})$ is the desired subspace by Theorem 4.7, being closed and bounded. $\diamond$

We can finally show that $\mathscr{G}$ is totally bounded. Indeed, we have seen there exists a compact subspace Y of $\mathbb{R}^n$ such that $\mathscr{G} \subset C^0(X,Y)$ and so, since $\mathscr{G}$ is equicontinuous, Proposition 4.4 implies that $\mathscr{G}$ is totally bounded. ■

The reader should notice that the only special property of $\mathbb{R}^n$ used in the proof of Theorem 4.13 is the fact that a closed and bounded subspace is compact and so the theorem still holds when substituting $\mathbb{R}^n$ with any metric space satisfying this property.

Exercise 4.5 Let $\mathscr{F} \subset C^0([0,1],\mathbb{R})$ be closed, bounded and equicontinuous. Show that there exists $g \in \mathscr{F}$ such that

$$\int_0^1 g(x)\,\mathrm{d}x \ge \int_0^1 f(x)\,\mathrm{d}x,$$

for every $f \in \mathscr{F}$.

Hint: Consider the function $G\colon C^0([0,1],\mathbb{R}) \to \mathbb{R}$ given by $G(f) = \int_0^1 f(x)\,\mathrm{d}x$. Enter ■

We finally state yet another version of Ascoli-Arzelà Theorem, this time involving sequential compactness:

Theorem 4.14 — Ascoli-Arzelà Theorem, third version. Let X be a compact topological space, $(\mathbb{R}^n, d)$ the Euclidean metric space and $\{f_n\}$ a sequence of functions in $C^0(X,\mathbb{R}^n)$. If the family $\mathscr{F} = \{f_n : n \in \mathbb{N}\}$ is pointwise bounded and equicontinuous, then the sequence $\{f_n\}$ has a convergent subsequence.

Proof. By Theorem 4.13, the closure of $\mathscr{F}$ is compact and so sequentially compact (Theorem 4.9). ■

■ **Example 4.8** Let $\{f_n\}$ be a sequence of functions in $C^1([0,1],\mathbb{R})$ such that, for every $n \in \mathbb{N}$,

$$|f_n'(x)| \le \frac{1}{\sqrt{x}} \quad (\text{for every } 0 < x \le 1) \quad \text{and} \quad \int_0^1 f_n(x)\,\mathrm{d}x = 0.$$

Then $\{f_n\}$ has a subsequence converging uniformly on $[0,1]$.

By Theorem 4.14, it is enough to show that $\mathscr{F} = \{f_n : n \in \mathbb{N}\}$ is pointwise bounded and equicontinuous. Let us begin with equicontinuity. For every $0 \leq x < y \leq 1$ and $n \in \mathbb{N}$, we have

$$|f_n(y) - f_n(x)| = \left| \int_x^y f_n'(t)\ \mathrm{d}t \right| \leq \int_x^y |f_n'(t)|\ \mathrm{d}t \leq \int_x^y \frac{1}{\sqrt{t}}\ \mathrm{d}t = 2\sqrt{y} - 2\sqrt{x}.$$

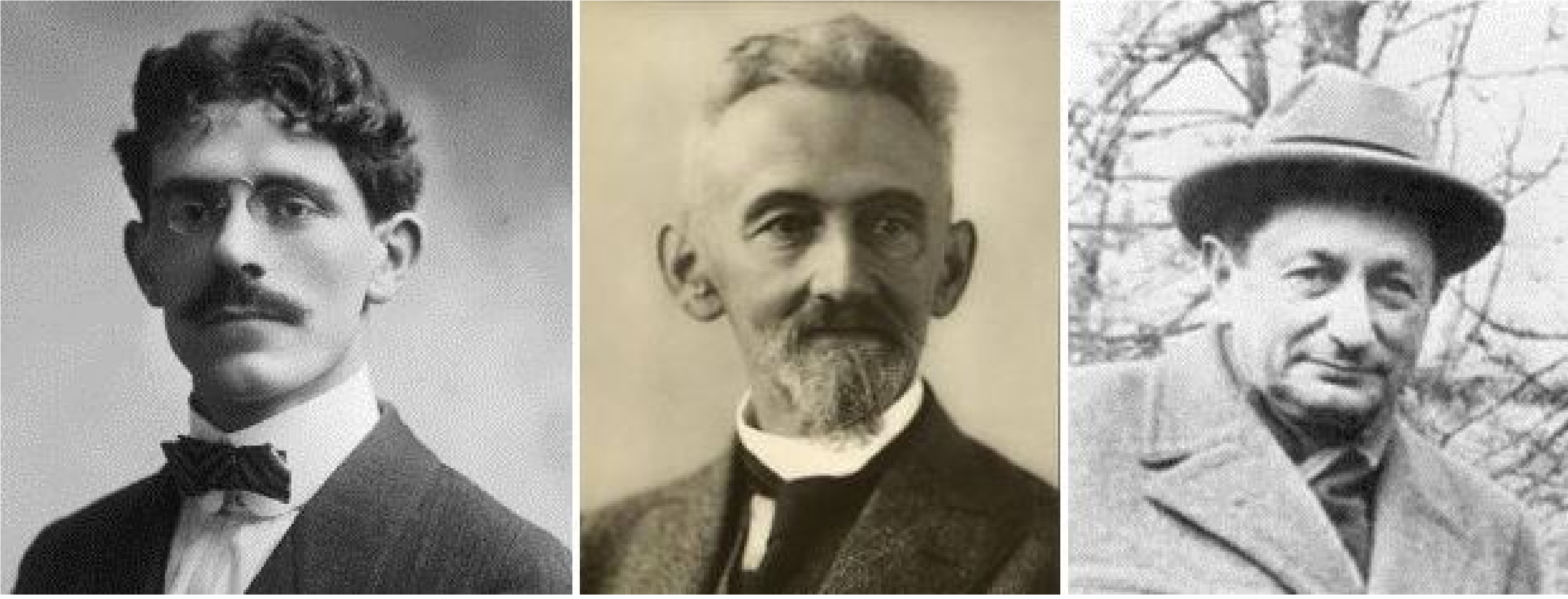
Since the function $F(x) = 2\sqrt{x}$ is continuous on the compact $[0,1]$, it is uniformly continuous (Theorem 4.8) and so, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|F(y) - F(x)| < \varepsilon$ for every $x, y \in [0,1]$ with $|y - x| < \delta$. The inequality above then implies $\mathscr{F}$ is equicontinuous.

As for pointwise boundedness, note that since $\int_0^1 f_n(x)\ \mathrm{d}x = 0$, there exists $z \in [0,1]$ such that $f_n(z) = 0$. Therefore, for every $x \in [0,1]$, we have $|f_n(x)| \leq 2|\sqrt{x} - \sqrt{z}| \leq 4$. ■

Exercise 4.6 Let $\{f_n\}$ be a family of functions in $C^0([0,1], \mathbb{R})$ such that there exists $M > 0$ with $|f_n(x)| < M$, for every $n \in \mathbb{N}$ and $x \in [0,1]$. Moreover, let $F_n \colon [0,1] \to \mathbb{R}$ given by

$$F_n(x) = \int_0^x f_n(t)\ \mathrm{d}t.$$

Show that the sequence $\{F_n\}$ has a subsequence converging uniformly on $[0,1]$. ■



5. Baire Spaces

In this chapter we introduce some notions defining a "topological size" of sets.

Definition 5.1 — Nowhere dense subset. A subset $S \subset X$ of a topological space X is nowhere dense in X if $\mathrm{int}(\overline{S}) = \varnothing$.

Nowhere dense subsets of a topological space can be somehow considered to be "very small".

Example 5.1 $\mathbb{Z}$ is nowhere dense in $\mathbb{R}$. The set $\{1/n : n \in \mathbb{N}\}$ is nowhere dense in $\mathbb{R}$. ■

Proposition 5.1 Let X be a topological space and $S \subset X$. The following are equivalent:
(i). S is nowhere dense, i.e. $\mathrm{int}(\overline{S}) = \varnothing$.
(ii). $\overline{S}$ contains no non-empty open subset.
(iii). $X \setminus \overline{S}$ is dense.

Proof. (i) $\Rightarrow$ (ii) It follows from the fact that $\mathrm{int}(\overline{S})$ is the union of all open sets contained in $\overline{S}$ (Proposition 2.2).

(ii) $\Rightarrow$ (iii) Since $\overline{S}$ contains no non-empty open subset, $X \setminus \overline{S}$ intersects every non-empty open subset and so it is dense (Proposition 2.3).

(iii) $\Rightarrow$ (i) Again by Proposition 2.3, every non-empty open subset intersects $X \setminus \overline{S}$ and so $\overline{S}$ has no interior points. ■

Lemma 5.1 Let X be a topological space. The union of finitely many nowhere dense sets in X is a nowhere dense set in X.

Proof. We just prove the statement for two nowhere dense sets A and B. The general case easily follows. Since $\overline{A \cup B} = \overline{A} \cup \overline{B}$ (Exercise 2.5), we have that $\mathrm{int}(\overline{A \cup B}) \subset \overline{A} \cup \overline{B}$. Therefore, $\mathrm{int}(\overline{A \cup B}) \cap (X \setminus \overline{B})$ is an open subset contained in $\overline{A}$ and so, since A is nowhere dense, it must be empty. But then $\mathrm{int}(\overline{A \cup B}) \subset \overline{B}$ and since B is nowhere dense, we have that $\mathrm{int}(\overline{A \cup B}) = \varnothing$. ■

Note that the union of countably many nowhere dense sets need not be nowhere dense. Indeed, the subset $\mathbb{Q}$ of $\mathbb{R}$ is a countable union of one-point sets (which are clearly nowhere dense) but it is not nowhere dense as $\operatorname{int}(\overline{\mathbb{Q}}) = \mathbb{R}$. This motivates the introduction of the following:

Definition 5.2 — Meager set, non-meager set, residual set. Let X be a topological space and A a subset of X. The subset A is meager in X if it can be written as a countable union of nowhere dense sets. A is non-meager if it is not meager. A is residual if it is the complement of a meager set.

It follows immediately from the definition that a subset of a meager set is meager and that a union of countably many meager sets is meager.

Meager sets are in some sense “small” and residual sets are in some sense “large” (i.e. their complements are “small”). However, note that non-meager sets are not necessarily large (they are just not “small”).

■ **Example 5.2** It is important to stress that a set is meager, non-meager or residual *in* a certain topological space. For example, $\mathbb{Z}$ is non-meager in itself. Indeed, every subset of $\mathbb{Z}$ is open and so there are no non-empty nowhere dense subsets. On the other hand, $\mathbb{Z}$ is meager in $\mathbb{R}$, being nowhere dense in $\mathbb{R}$. ■

■ **Example 5.3** The set S of points with at least one rational coordinate in $\mathbb{R}^2$ is meager. Indeed, if $\{q_n\}$ is an enumeration of $\mathbb{Q}$, we have that the lines $A_n = \{(q_n, x) : x \in \mathbb{R}\}$ and $B_n = \{(x, q_n) : x \in \mathbb{R}\}$ are nowhere dense. But then $S = \bigcup A_n \cup \bigcup B_n$ is meager. ■

Definition 5.3 — Baire space. A topological space X is a Baire space if every meager set in X has empty interior.

Proposition 5.2 Let X be a topological space. The following are equivalent:

(i). X is a Baire space, i.e. every meager set in X has empty interior.
(ii). Every non-empty open set in X is non-meager.
(iii). Every residual set in X is dense in X.
(iv). The union of countably many closed sets of X with empty interior in X has empty interior in X.
(v). The intersection of countably many open subsets of X which are dense in X is dense in X.

Proof. (i) $\Rightarrow$ (ii) This is obvious.

(ii) $\Rightarrow$ (iii) Let S be a residual set in X and suppose that S is not dense in X. By Proposition 2.3, $\operatorname{int}(X \setminus S)$ is a non-empty open set and so, by assumption, it is not meager. But this contradicts the fact that $\operatorname{int}(X \setminus S)$ is a subset of the meager set $X \setminus S$.

(iii) $\Rightarrow$ (iv) Let $\bigcup C_n$ be a union of countably many closed subsets of X with empty interior in X. Clearly, each C_n is nowhere dense in X and so $\bigcup C_n$ is a meager set. By assumption, the residual set $X \setminus \bigcup C_n$ is dense and so $\operatorname{int}(\bigcup C_n) = \varnothing$.

(iv) $\Rightarrow$ (v) Let $\bigcap U_n$ be an intersection of countably many open subsets of X which are dense in X. Therefore, $X \setminus U_n$ has empty interior in X, for each n. But then $X \setminus \bigcap U_n = \bigcup X \setminus U_n$ is a union of countably many closed subsets of X with empty interior and so, by assumption, $X \setminus \bigcap U_n$ has empty interior, i.e. $\bigcap U_n$ is dense.

(v) $\Rightarrow$ (i) Let S be a meager set, i.e. $S = \bigcup A_n$ is a countable union of nowhere dense sets A_n. Therefore, $X \setminus \bigcup \overline{A_n} = \bigcap X \setminus \overline{A_n}$ is an intersection of countably many open subsets of X which are dense in X and so, by assumption, $X \setminus \bigcup \overline{A_n}$ is dense. This means that $\mathrm{int}(\bigcup \overline{A_n}) = \varnothing$ and the conclusion follows from the fact that $S \subset \bigcup \overline{A_n}$. ■

In this chapter, we provide two sufficient conditions for a space to be Baire. We begin by showing that every complete metric space is Baire, which is one version of the celebrated Baire Category Theorem. In order to do so, we first need a useful result on decreasing sequences of non-empty closed sets in a topological space X, namely sequences $\{C_n\}$ such that $C_1 \supset C_2 \supset \cdots$.

Clearly, for every finite decreasing sequence, the intersection $\bigcap C_n$ is non-empty. By using the formulation of compactness in terms of the finite intersection property (Theorem 4.4), it is easy to see that if X is compact, the intersection is non-empty. However, this is not true in general. For example consider $X = \mathbb{R}$ and $C_n = [n, +\infty)$ or $X = \mathbb{Q}$ and $C_n = \mathbb{Q} \cap [\sqrt{2} - 1/n, \sqrt{2} + 1/n]$. The following result tells us that these are essentially the only two obstructions to a non-empty intersection:

Theorem 5.1 — Cantor's Intersection Theorem. Let (X, d) be a complete metric space and let $\{C_n\}$ be a sequence of non-empty closed subsets of X such that $C_1 \supset C_2 \supset \cdots$ and $\lim_{n \to \infty} \mathrm{diam}(C_n) = 0$. Then $\bigcap C_n$ contains precisely one point of X.

Proof. For each $n \in \mathbb{N}$, let $x_n \in C_n$. We show that the sequence $\{x_n\}$ is Cauchy. Therefore, let $\varepsilon > 0$. Since $\{\mathrm{diam}(C_n)\}$ converges to 0, there exists n_ε such that $\mathrm{diam}(C_n) < \varepsilon$, for every $n \geq n_\varepsilon$. Let now $n, m \geq n_\varepsilon$. Since $C_1 \supset C_2 \supset \cdots$, we have that $x_n, x_m \in C_{n_\varepsilon}$ and so $d(x_n, x_m) \leq \mathrm{diam}(C_{n_\varepsilon}) < \varepsilon$. Therefore, $\{x_n\}$ is a Cauchy sequence in X and by completeness it converges to some $x \in X$. Observe now that, for a fixed k, the subsequence $\{x_i\}_{i \geq k}$ converges to x and so, since C_k is closed, Lemma 1.4 implies that $x \in C_k$. Therefore, $x \in \bigcap C_n$.

Let us now show that in fact $\bigcap C_n = \{x\}$. Suppose, to the contrary, there exists $x' \neq x$ with $x' \in \bigcap C_n$. Let $\varepsilon = d(x, x') > 0$ and choose n sufficiently large such that $\mathrm{diam}(C_n) < \varepsilon$. Since $x, x' \in C_n$, we have $d(x, x') \leq \mathrm{diam}(C_n) < \varepsilon = d(x, x')$, a contradiction. ■

We now prove Baire Category Theorem by playing the following topological game!

Definition 5.4 — Choquet game. The Choquet game is a (infinite) two-player game played in a given metric space X as follows. Antoine moves first by choosing a non-empty open set $U_1 \subset X$. Then Bertrand moves by choosing a non-empty open set $V_1 \subset U_1$. Antoine then chooses a non-empty open set $U_2 \subset V_1$, etc. This gives two decreasing sequences $\{U_n\}$ and $\{V_n\}$ of non-empty open sets with $U_n \supset V_n \supset U_{n+1}$, for each n, and $\bigcap U_n = \bigcap V_n$. Antoine wins if $\bigcap U_n = \varnothing$ and Bertrand wins if $\bigcap U_n \neq \varnothing$.

Theorem 5.2 — Baire Category Theorem, first version. If X is a complete metric space, then it is a Baire space.

Proof. We prove the theorem with the aid of the Choquet game played by Antoine and Bertrand in an arbitrary metric space X. We say that a player has a winning strategy if he has a method allowing him to win no matter what the opponent does. The statement is immediately obtained from the following two claims:

If X is complete, then Bertrand has a winning strategy.

Indeed, suppose (X, d) is complete. Let $\overline{B}_\varepsilon(x)$ denote the closed ball centered at x with radius ε. After each play U_i of Antoine, Bertrand simply chooses a non-empty open ball $B_{1/n_i}(x_i) \subset U_i$, with $n_i \in \mathbb{N}$, such that $\overline{B}_{1/n_i}(x_i) \subset U_i$ (it is easy to see such a ball indeed exists). But then we have a decreasing sequence $\overline{B}_{1/n_1}(x_1) \supset \overline{B}_{1/n_2}(x_2) \supset \cdots$ of closed balls such that $\lim_{i\to\infty} \operatorname{diam}(\overline{B}_{1/n_i}(x_i)) = 0$. Therefore, by Theorem 5.1, $\varnothing \neq \bigcap \overline{B}_{1/n_i}(x_i) \subset \bigcap U_i$. $\diamond$

If X is not Baire, then Antoine has a winning strategy.

Indeed, if X is not Baire, there exists a countable family $\{C_n\}$ of closed subsets of X having empty interior in X such that their union $\bigcup C_n$ has non-empty interior in X (Proposition 5.2). Therefore, there exists a non-empty open subset U of X contained in $\bigcup C_n$. Antoine then proceeds as follows. First, he chooses $U_1 = U$. At the $n+1$-th play, he chooses $U_{n+1} = V_n \setminus C_n$, where V_n is the n-th play of Bertrand. It is easy to see this is indeed a legal move, i.e. that U_{n+1} is a non-empty open set contained in V_n. Using this strategy, Antoine forces the plays of Bertand to satisfy: $V_1 \subset U_1 = U$ and $V_{n+1} \subset V_n \setminus C_n$, for every $n \geq 1$. But then $\bigcap U_n = \bigcap V_n \subset U \setminus \bigcup C_n = \varnothing$. $\diamond$ ■

The Baire Category Theorem provides an extremely powerful device for showing that something exists (or not).

Corollary 5.1 A complete metric space is not a countable union of nowhere dense sets.

The previous corollary gives a proof of the fact that $\mathbb{R}$ is uncountable. Indeed, $\mathbb{R}$ is complete and $\{x\}$ is nowhere dense for each $x \in \mathbb{R}$.

■ **Example 5.4** Consider the Euclidean metric space $\mathbb{R}$. The subset $\mathbb{Q}$ cannot be written as a countable intersection of open sets.

Indeed, suppose this is not the case. Then $\mathbb{R}\setminus\mathbb{Q}$ is a countable union of closed sets and since $\mathbb{Q}$ is clearly a countable union of closed sets as well, we have that $\mathbb{R}$ is a countable union of closed sets such that each of them belongs to either $\mathbb{Q}$ or $\mathbb{R}\setminus\mathbb{Q}$. On the other hand, by Baire Category Theorem, one of these sets has non-empty interior, a contradiction. ■

We now show a first application of Baire Category Theorem. Recall that a family of functions $\mathscr{F}$ from a topological space X to the Euclidean metric space $\mathbb{R}$ is pointwise bounded if, for every $x \in X$, the set $\mathscr{F}_x = \{f(x) : f \in \mathscr{F}\}$ is bounded (under the Euclidean metric), i.e. there exists $M_x > 0$ such that $|f(x)| \leq M_x$ for every $f \in \mathscr{F}$. If we can find a bound M independent of x, we obtain the following notion:

Definition 5.5 — Uniformly bounded family. Let X be a topological space and $\mathscr{F}$ a family of functions $f\colon X \to \mathbb{R}$. The family $\mathscr{F}$ is uniformly bounded on X if there exists $M > 0$ such that $|f(x)| \leq M$ for every $f \in \mathscr{F}$ and $x \in X$.

■ **Example 5.5** Given an enumeration $q_1, q_2, \ldots$ of $\mathbb{Q}$, we define a function $f_n\colon \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} k & \text{if } x = q_k \text{ and } n \leq k; \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

The family $\mathscr{F} = \{f_n : n \in \mathbb{N}\}$ is pointwise bounded but not uniformly bounded. Pointwise boundedness is evident. On the other hand, since $\mathbb{Q}$ is dense in $\mathbb{R}$ no function f_n is bounded on any interval (a,b). ■

Theorem 5.3 — Uniform boundedness principle. If X is a complete metric space and $\mathscr{F} \subset C^0(X,\mathbb{R})$ a pointwise bounded family of continuous functions, then there exists a non-empty open set U of X on which $\mathscr{F}$ is uniformly bounded.

Proof. Since f is continuous, $f^{-1}([-n,n])$ is closed in X, for each $n \in \mathbb{N}$. Therefore, $C_n = \bigcap_{f\in\mathscr{F}} f^{-1}([-n,n])$ is closed as well. On the other hand, since $\mathscr{F}$ is pointwise bounded, we have that, for each $x \in X$, there exists $M_x > 0$ such that $f(x) \in [-M_x, M_x]$ for every $f \in \mathscr{F}$. This implies that each $x \in X$ belongs to some C_n and so $X = \bigcup_{n\in\mathbb{N}} C_n$. But since X is Baire (being complete), there exists C_k with non-empty interior and so C_k contains an open set U. Therefore, $|f(x)| \leq k$ for every $f \in \mathscr{F}$ and $x \in U \subset C_k$. ■

We now give yet another application of Baire Category Theorem. Note that $\mathbb{R}^n$, although not compact, can be written as a countable union of compact sets. It turns out this is not the case of $C^0([0,1],\mathbb{R})$:

Proposition 5.3 $C^0([0,1],\mathbb{R})$ is not a countable union of compact sets.

Proof. The proof nicely combines several results. Suppose $C^0([0,1],\mathbb{R}) = \bigcup K_n$ is a countable union of compact sets K_n. Since each compact in a metric space is closed (Lemma 4.2) and $C^0([0,1],\mathbb{R})$ is complete (Theorem 3.1), Baire Category Theorem implies there exists K_n with non-empty interior. Therefore, K_n contains an open ball $B_\varepsilon(g)$. Moreover, the closure of $B_\varepsilon(g)$ in K_n is compact (Lemma 4.1). But we know that in a normed space the closure of an open ball is just the corresponding closed ball (Exercise 2.4), and let us denote it by $\overline{B}_\varepsilon(g)$. Recall now that the closed ball in $C^0([0,1],\mathbb{R})$ centered at 0_f (the identically 0 function) with radius 1 is not compact. On the other hand, it is easy to see that $\overline{B}_\varepsilon(f)$ is homeomorphic to $\overline{B}_1(0_f)$ and since compactness is invariant under homeomorphisms (Lemma 4.3), we obtain a contradiction. ■

Does there exist a function $f\colon \mathbb{R} \to \mathbb{R}$ which is continuous at the rationals and discontinuous at the irrationals? We briefly sketch how a negative answer can be obtained from the Baire Category Theorem. The reader is invited to work out the details.

For every open interval I of $\mathbb{R}$, define the oscillation of f over I as $\omega(f,I) = \sup_{x\in I} f(x) - \inf_{x\in I} f(x)$. Moreover, for $x \in \mathbb{R}$, the oscillation of f at x is $\omega(f,x) = \inf_{I\ni x} \omega(f,I)$. It is not difficult to see that f is continuous at x if and only if $\omega(f,x) = 0$. Consider now the set S of points of $\mathbb{R}$ at which f is not continuous and let $S_n = \{x \in \mathbb{R} : \omega(f,x) \geq 1/n\}$. We have that each S_n is closed and $S = \bigcup_{n\geq 1} S_n$. But then the reasoning in Example 5.4 implies that S cannot be $\mathbb{R}\setminus\mathbb{Q}$.

However, there exists a function $f\colon \mathbb{R} \to \mathbb{R}$ which is continuous at the irrationals and discontinuous at the rationals. One famous example is given by the following, known as Thomae's function:

$$f(x) = \begin{cases} \left|\dfrac{1}{q}\right| & \text{if } x \in \mathbb{Q} \text{ and } x = \dfrac{p}{q} \text{ with } \gcd(p,q) = 1; \\ 0 & \text{if } x \in \mathbb{R}\setminus\mathbb{Q}. \end{cases}$$

We conclude the chapter with another version of Baire Category Theorem:

Theorem 5.4 — Baire Category Theorem, second version. If X is a compact Hausdorff space, then it is a Baire space.

In order to prove it, we need the following:

Proposition 5.4 Let X be a compact Hausdorff topological space, C a closed subset of X and $x \in X\setminus C$. If U is an open neighbourhood of x not contained in C, then there exists an open neighbourhood V of x such that $\overline{V} \subset U \setminus C$.

Proof. We first show there exist two disjoint open subsets V and W of X such that $x \in V$ and $C \cup (X\setminus U) \subset W$. Let $B = C \cup (X\setminus U)$ and observe that $x \notin B$. Since X is Hausdorff, for every $y \in B$, there exist disjoint open subsets V_y and W_y such that $x \in V_y$ and $y \in W_y$. Moreover, since B is a closed subset of the compact space X, it is compact (Lemma 4.1). This implies that the open covering $\{W_y \cap B\}_{y \in B}$ of B (in the subspace topology) admits a finite subcovering. Therefore, there exist $W_{y_1}, \dots, W_{y_n}$ such that $B \subset \bigcup_{i=1}^n W_{y_i}$. But $W = \bigcup_{i=1}^n W_{y_i} \supset B$ is an open set, $V = \bigcap_{i=1}^n V_{y_i}$ is an open set containing x and the two are disjoint, as desired.

Let us now show that $\overline{V} \subset U \setminus C$. We have

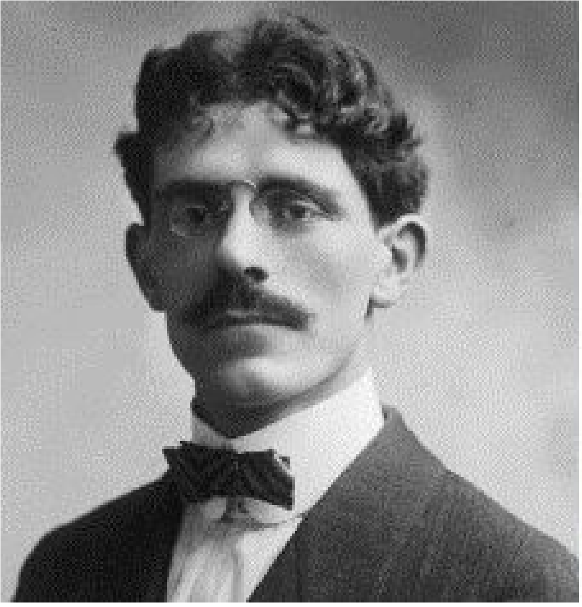
$$V \subset X \setminus W \subset X \setminus (C \cup (X \setminus U)) = X \setminus C \cap X \setminus (X \setminus U) = X \setminus C \cap U = U \setminus C.$$

Moreover, since $X \setminus W$ is a closed subset of X containing V, we have that $\overline{V} \subset X \setminus W$ (Proposition 2.2). Therefore, $x \in V \subset \overline{V} \subset X \setminus W \subset U \setminus C$, as desired. ■

Proof of Theorem 5.4. We show that, given a countable family $\{C_n\}$ of closed sets of X with empty interiors, their union $\bigcup C_n$ has empty interior in X. In other words, we show that each open set of X contains a point not contained in any C_i.

Therefore, let U_0 be an arbitrary open set of X. We recursively construct a sequence of open sets as follows. Since C_1 has empty interior, there exists $y_0 \in U_0 \cap (X \setminus C_1)$. By Proposition 5.4, there exists an open neighbourhood U_1 of y_0 such that $\overline{U_1} \subset U_0 \setminus C_1$. Given the non-empty open subset U_{n-1}, there exist $y_{n-1} \in U_{n-1} \cap (X \setminus C_n)$ (as every C_i has empty interior) and an open neighbourhood U_n of y_{n-1} such that $\overline{U_n} \subset U_{n-1} \setminus C_n$ (again by Proposition 5.4).

Observe now that $\bigcap \overline{U_n} \neq \varnothing$. Indeed, since $\overline{U_1} \supset \overline{U_2} \supset \cdots$, the family $\{\overline{U_n}\}$ of closed subsets of X has the finite intersection property and so, since X is compact, Theorem 4.4 implies that $\bigcap \overline{U_n} \neq \varnothing$. But then, if $x \in \bigcap \overline{U_n}$, we have that $x \in \overline{U_1} \subset U_0$ and since $\overline{U_n} \cap C_n = \varnothing$ for each n, we have $x \notin \bigcup C_n$. ■



Bibliography

[1] *17 Camels Trick*. Math Overflow. https://mathoverflow.net/questions/271608/17-camels-trick (cited on page 43).

[2] Keith Conrad. *The Contraction Mapping Theorem*. http://www.math.uconn.edu/~kconrad/blurbs/analysis/contraction.pdf (cited on page 40).

[3] Theodore W. Gamelin and Robert Everist Greene. *Introduction to Topology*. 2nd edition. Dover Publications, 1999.

[4] Steven G. Krantz. *Mathematical Apocrypha: Stories and Anecdotes of Mathematicians and the Mathematical*. The Mathematical Association of America, 2002 (cited on page 43).

[5] Mark de Longueville. *A Course in Topological Combinatorics*. Springer, 2013 (cited on page 39).

[6] Jiří Matoušek. *Lectures on Discrete Geometry*. Springer, 2002 (cited on page 46).

[7] James R. Munkres. *Topology*. 2nd edition. Prentice Hall, 2000.

[8] Charles C. Pugh. *Real Mathematical Analysis*. Springer, 2002 (cited on page 39).

[9] Volker Runde. *A Taste of Topology*. Springer, 2005.

[10] Christian Schnell. *Lecture Notes*. https://www.math.stonybrook.edu/~cschnell/pdf/notes/mat530.pdf (cited on page 39).

[11] Edoardo Sernesi. *Geometria 2*. Bollati Boringhieri, 1994.

[12] Joel H. Shapiro. *A Fixed-Point Farrago*. Springer, 2016 (cited on page 40).